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A general approach to Lehmann-Suwa-Khanedani index theorems: partial holomorphic connections and extensions of foliations

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Chapter 0

Introduction

Motivation

The topic of this thesis is localization of characteristic classes. One of the first examples of characteristic class is the **Euler class** of a rank l orientable vector bundle E on a smooth manifold M (we refer to [11] for the formal definition and its properties); the Euler class $e(E)$ is a cohomology class in $H^l(M,\mathbb{Z})$ which measures the extent to which the vector bundle is "twisted", i.e. measures the deviation of the local product structure from a global product structure. In particular, the Euler class vanishes when the vector bundle admits a global non-zero section. Suppose now we have a section v of E on M; we say that a point p in M is a singular point of v if v vanishes at p. We denote the set of singular points of v by Σ. What happens now is that, if we restrict E to $M \setminus \Sigma$ we have that on this set E admits a non-zero section. Therefore the Euler class of E restricted to $M \setminus \Sigma$ is 0. An important property of cohomology is that the pair $(M, M \setminus \Sigma)$ gives rise to a long exact sequence in cohomology

$$\cdots \longrightarrow H^l(M, M \setminus \Sigma) \xrightarrow{\iota} H^l(M) \xrightarrow{r} H^l(M \setminus \Sigma) \longrightarrow \cdots$$

where by ι we mean the extension by 0 and by r the restriction to a subset; since this sequence is exact and $r(e(E)) = 0$ we can lift the Euler class to a class in $H^l(M, M \setminus \Sigma)$; this is called a localization of $e(E)$ to Σ and we denote it by $e(E, v)$.

In Section 1.2 we will present two important results, the Alexander and the Poincaré dualities. Those results express an important relation between the cohomology and the homology of a compact connected orientable manifold M of dimension n; the Poincaré duality P_M shows us how to associate isomorphically to each class in $H^k(M;\mathbb{Z})$ a class in $H_{n-k}(M;\mathbb{Z})$ while the Alexander duality A_Σ shows us how to associate to each class in the relative cohomology $H^k(M, M \setminus \Sigma;\mathbb{Z})$ a class in the homology $H_{n-k}(\Sigma;\mathbb{Z})$. Moreover, if we take into account the inclusion map i^* induced in homology by $i : \Sigma \hookrightarrow M$ we have that the two operations commute; indeed, if ω is a class in the relative cohomology of the pair $(M, M \setminus \Sigma)$ we have that $P_M \circ \iota(\omega) = i^* A_\Sigma(\omega)$.

What happens now if we talk about the Euler class $e(TM)$ of the tangent bundle of M, supposing there exists a smooth vector field v which vanishes only at isolated points? We can lift $e(TM)$ to a class in $H^n(M, M \setminus \Sigma)$; then

$i^*A_\Sigma(e(TM,v)) = P_M(e(TM))$. Since M is connected, compact and orientable we have that $P_M(e(TM))$ is an integer, an important topological invariant of the manifold called the **Euler characteristic** $\chi(M)$; but what if Σ is decomposed in its connected components Σ_α? Then its homology can be seen as the direct sum of the homology of its connected components; thanks to the excision principle we have then that

$$\chi(M) = \sum_\alpha \iota^*(A_{\Sigma_\alpha} e(TM,v)).$$

On each connected component the Alexander duality associates to $e(TM,v)$ an element of $H_0(\Sigma_\alpha;\mathbb{Z})$, since Σ_α is connected this is an integer, called an **index**.

We refer to [31] for a formalization of this argument; anyway, this index depends on the local behaviour of the vector field near the singular point and is called the **Poincaré-Hopf index** of a vector field at the singularity. If M is a compact orientable surface, the Euler characteristic is related to the genus g of the surface by the equation $\chi(M) = 2-2g$. Therefore we have a condition that needs to be satisfied by a smooth vector field on a surface. As an example suppose the genus of M is 0; then the sum of the Poincaré-Hopf indices of a vector field must be 2; if v has no singularities on M, this condition can not be satisfied by v. So, on a genus 0 surface, e.g., a sphere, every vector field must have at least one singularity and we get important informations on the possible dynamics on a sphere by purely topological constraints. It is clear that the important and powerful tool that permitted us to prove this result was nothing else that the localization of the Euler class. The Euler class is only one of several characteristic classes that we can associate to a vector bundle ([27]); in my thesis I will deal with the localization of Chern classes associated to some particular coherent sheaf.

The Chern classes are cohomological invariants of a smooth complex vector bundle [11]; it is possible to give many different definitions of them, e.g. using universal bundles [11] or obstruction theory [30]; we use the differential geometric framework of Chern-Weil theory.

In this thesis I deal with localization of Chern classes arising from the existence of foliations; this topic stems from the foundational papers of Raoul Bott [8] and [9], in which the Bott Vanishing Theorem was first established. This theorem says that if there exists a “holomorphic action” of a vector field on a holomorphic bundle E over a complex n-dimensional manifold M, given a symmetric polynomial ϕ of degree n, then the characteristic class $\phi(E)$, obtained by evaluating ϕ on the Chern classes of E, vanishes. This vanishing theorem gives rise to a localization process and to what are called Baum-Bott residues.

In Chapter 2 I give an account of the theory behind Baum-Bott residues and explain how to compute the residue associated with the isolated singularity of a holomorphic vector field, following [31]. An example of application is the proof of the existence of topological obstructions to integrability; an example of this phenomenon is the fact that $T\mathbb{CP}^n$ has a holomorphic subbundle of codimension 1 but no integrable holomorphic subbundle of codimension 1. A proof of this fact can be found, e.g., in [10]. So, everything seems to lead to the fact that localization of characteristic classes is an important tool in differential geometry, topology and dynamics. In particular, the existence of the Bott Vanishing Theorem stresses the importance of localization of Chern classes for complex dynamical systems.

Another example of the importance of residues in complex dynamical systems is paper [16] where Camacho and Sad used the computation of a residue to build up a combinatorial argument proving the existence of a complex separatrix passing through a singularity of a complex vector field on a complex surface.

Even if at a first sight this index theorem didn't seem directly connected to localization of characteristic classes a global framework for this theory was later given by Lehmann and Suwa [31], including also this index in the theoretical account of localization of characteristic classes. The fundamental principle ([4]) is that the existence of a flat partial holomorphic connection (analogous to the holomorphic action of Bott) implies the vanishing of the Chern classes associated to a vector bundle. The point is that the existence of a holomorphic foliation $\mathcal{F}$ leaving a submanifold S invariant gives rise to partial holomorphic connections on $N_{\mathcal{F}|_S}$, the normal bundle of the foliation seen as a quotient of the tangent bundle of the submanifold (Baum-Bott index), N_S, the normal bundle to the submanifold (Camacho-Sad index) and $N_{\mathcal{F}}|_S$ the normal bundle of the foliation seen as a quotient of the ambient tangent bundle restricted to S (Khanedani-Lehmann-Suwa index). In particular this last type of theorems are of interest for my thesis: the existence of such an index theorem was first established in [22] and then developed in [26].

Now, at least two different research directions arise: to adapt such a theory to singular manifolds and submanifolds [25], and to try to develop methods to find new partial holomorphic connections and therefore new vanishing theorems. Indeed, many new vanishing theorems have appeared in literature and the topic of index theorems arising from foliations, which seemed reserved to the treatment of continuous holomorphic dynamics, found an important application also in the study of discrete holomorphic dynamics. In his paper [1] Abate finds an index theorem for holomorphic self-maps and uses it to prove an analogous of the Leau-Fatou flower theorem for holomorphic self-maps of $\mathbb{C}^2$ tangent to the identity. The results of this paper were later generalized in [3] and opened a whole new field of research.

The research into the topic of index theorems has been flourishing during the last years, with many works dealing with the case of foliation transverse to a submanifold in the Camacho-Sad and Baum-Bott case [4], [13], [14], [15].

The key to the existence of partial holomorphic connections is the vanishing of the Atiyah class, a cohomological obstruction to the splitting of a short exact sequence of sheaves of $\mathcal{O}_S$-modules [6]. In my work I follow the line of paper [4], where the Atiyah sheaf for the normal bundle of a submanifold was described in a more concrete way, giving new insights to the problem. Further developments as [5] showed the strong connection between the existence of partial holomorphic connections for N_S and the "regularity" of the embedding of a subvariety.

Results

In this thesis I prove an extension of the vanishing theorem for the Khanedani-Lehmann-Suwa action, definining and using what I call **foliations of the infinitesimal neighborhoods** of a submanifold S (Section 4.1); foliations of infinitesimal neighborhoods are a natural generalization of foliations and in I prove a Frobenius-type theorem.

Theorem (Frobenius Theorem for k-th infinitesimal neighborhoods, Chapter 4 Theorem 4.1.4)**.** *Suppose S is a non singular complex submanifold of codimension m in a complex manifold M of dimension n and suppose we have a foliation $\mathcal{F}$ of $S(k)$ of rank l. Then there exists an atlas $\{U_\alpha, \phi_\alpha\}$adapted to S such that if $U_\alpha \cap U_\beta \cap S \neq \emptyset$ then:*

$$\left[\frac{\partial z_\alpha^t}{\partial z_\beta^i}\right]_{k+1} = 0,$$

for $t = 1, \ldots, m, m+l+1, \ldots, n$ and $i = m+1, \ldots, m+l$ on $U_\alpha \cap U_\beta$.

Moreover I prove that their existence gives rise to the vanishing of the Atiyah class, which in turn permits me to prove an extension of the variation action (I state here the most general form and refer to Section 5.1 and Section 5.2).

Proposition (Chapter 5, Proposition 5.2.3)**.** *Suppose $\mathcal{E}$ is a coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$, S-faithful. Then there exists a partial holomorphic connection $(\delta, \mathcal{E})$ for $\mathcal{N}_{\mathcal{F},M}$.*

This permits me to prove a Lehmann-Suwa-Khanedani type index theorem.

Theorem (Chapter 5 Theorem 5.4.1)**.** *Let S be a codimension m compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a rank l foliation on S, such that it extends to the first infinitesimal neighborhood of $S \setminus S(\mathcal{F})$, and let $S(\mathcal{F}) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $S(\mathcal{F})$ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n-m-l$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F},M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{F,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

I prove also a Kahnedani-Lehmann-Suwa-type vanishing theorem under weaker conditions.

Theorem (Chapter 5 Theorem 5.4.2)**.** *Let S be a codimension m compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a foliation on S and let $\mathcal{E}$ be a rank l subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$. Suppose moreover that it is S-faithful. Let $\Sigma = S(\mathcal{F}) \cup S(\mathcal{E})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{E}, \mathcal{N}_{F,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{F,M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{E}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{F,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

The problem becomes then to study how to build foliations of the first infinitesimal neighborhood or to extend a foliation on S to the first infinitesimal neighborhood. I study under which conditions I can project a transversal foliation to a foliation of the first infinitesimal neighborhood.

Lemma (Chapter 4 Lemma 4.3.4)**.** *Let M be a n-dimensional complex manifold, S a submanifold of codimension r. Sequence*

$$0 \to \mathcal{T}_{S(1)} \to \mathcal{T}_{M,S(1)} \to \mathcal{N}_{S(1)} \to 0,$$

splits if S is 2-splitting, i.e., if there exists an atlas adapted to S such that:

$$\left[\frac{\partial z_\beta^p}{\partial z_\alpha^r}\right]_2 \equiv [0]_2,$$

for $p = m+1, \dots, n$ and $r = 1, \dots, m$.

In case S is splitting, I found a cohomological obstruction to the extension (which may not be involutive) of a foliation to the first infinitesimal neighborhood of S. This is an interesting point since it identifies some of the cohomological obstructions to the extension of foliations to a neighborhood of a submanifold.

Proposition (Chapter 3 Proposition 4.4.3)**.** *Let M be a complex manifold of dimension n, and S a splitting codimension m submanifold. Let $\mathcal{F}$ be a foliation of S and $\pi : N_S \to M$ the normal bundle of S in M. Let $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ and $\mathcal{V}$ the vertical foliation given by $\ker d\pi$. The sequence:*

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{pr} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits if there exists an atlas adapted to $\mathcal{F}$ and S such that

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^i \partial z_\beta^s} \in \mathcal{I}_S,$$

for all $r, s = 1, \dots, m$ and $i = m+1, \dots, m+l$.

This permits me to prove some index theorems for foliations and holomorphic self-maps, also in the transversal case (these results are collected in Section 5.4, Section 6.3). I state a couple of these theorems, to give an idea of the results obtained.

Theorem. *Let S be a codimension m 2-splitting compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a rank l holomorphic foliation defined on a neighborhood of S. Suppose there is a 2-splitting first order $\mathcal{F}$-faithful outside an analytic subset Σ of U containing $S(\mathcal{F}) \cap S$ and that S is not contained in Σ. Let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k bigger than $n-m-l$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F}^\sigma,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F}^\sigma,M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F}^\sigma,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F}^\sigma,M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}^\sigma,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}^\sigma,M}$.

Theorem. *Let S be a compact codimension m submanifold 2-splitting in M, an n dimensional complex manifold. Let $f \in End(M, S)$, $f \neq id_M$, $\nu_f > 1$. Suppose we have a splitting $\mathcal{F}_f$-faithful outside an analytic subset of S containing $S(\mathcal{F}_f)$. Let now $\mathcal{G}$ be the involutive closure of $\mathcal{F}_f^\sigma$ and $\Sigma = S(\mathcal{F}_f^\sigma) \cup S(\mathcal{G})$; let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be its decomposition in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f^\sigma$ and $\mathcal{N}_{\mathcal{G},M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{G},M}),$$

where $\phi(\mathcal{N}_{\mathcal{G},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{G},M}$.

I also compute the respective indices (the computations are in Section 5.5, Section 5.6, Section 6.4).

Plan of the Thesis

In Chapter 1 I present the background material which is needed for the understanding of the methods involved in my results. It starts in Section 1.2 with the proof of Poincaré and Alexander dualities. As shown in our discussion about the Poincaré-Hopf index theorem these two main results permit us to define residues and indices. Section 1.3 introduces the tool of Čech cohomology and the concept of sheaf; this cohomology permits us to deal easily with the problem of the existence of global sections of a sheaf and all the connected questions. In Section 1.4 I define an important tool, Čech-de Rham cohomology; this cohomology, isomorphic to the de Rham cohomology, is used since together with its integration theory presented in Section 1.5 it seems to be the most useful framework when dealing with the residue problem. In the following Section 1.6 the Chern-Weil theory of Chern classes is presented; I define moreover an important geometrical object, the Bott Difference Form which is used throughout the paper and takes care of the ambiguities in the definition of the Chern classes arising from the choice of different connections; after I have defined the Chern classes I adapt the theory developed for them to Čech-de Rham cohomology in Section 1.7. Section 1.8 is devoted to precise two main objects of our study, i.e. foliations and coherent sheaves and some of their properties. In section 1.9 I present a general principle that is used many times: a short exact sequence of coherent sheaves splits if and only if an associated class in cohomology vanishes. One of the first applications of this general principle is to study the regularity of the embedding of a submanifold S in a complex manifold M. In Section 1.10 we define what a splitting manifold is and prove some results concerning it and its generalizations.

Chapter 2 presents the Bott Vanishing Theorem and the Baum-Bott type residues. In Section 2.1 a proof of Bott's Theorem is given, in the form developed in [4], which applies also to non involutive subbundles. Section 2.2 is devoted to explain the basis of the localization principle, using as an example the Lehmann-Khanedani-Suwa index theorem. Then Sections 2.3 and 2.4 present the Baum-Bott index theorems and the computation of the residue in the case of an isolated singularity of a holomorphic vector field. This complicated computation is interesting for many reasons: first, it shows that a lot of work has to be done to compute the residue even in the simplest cases and is an interesting model for the computation of other residues.

Chapter 3 is devoted to Atiyah's theory of holomorphic connections. In Section 3.1 is given a proof of the fact that the splitting of the partial Atiyah sequence for a bundle E gives rise to a partial holomorphic connection for E. Given a foliation $\mathcal{F}$ of a submanifold S Section 3.2 is devoted to the computation of the Atiyah class for the quotient $\mathcal{N}_{\mathcal{F},M} := \mathcal{T}_{M,S} \,/\, \mathcal{F}$. In Section 3.3 I define what I mean for tangential sheaf for the infinitesimal neighborhood of a submanifold and prove that this sheaf admits a well defined bracket operation; these concepts are then used in Section 3.4 to define a more concrete realization for the Atiyah sheaf of $\mathcal{N}_{\mathcal{F},M}$.

Chapter 4 contains the core of the thesis. In Section 4.1 I define foliations of infinitesimal neighborhoods and prove a Frobenius type result for such foliations: the existence of an atlas for the ambient manifols where the transition functions for the tangent bundle have a special form. Then, some work is done to precise the objects we are working with: in Section 4.2 I define what we mean by singular foliations of the infinitesimal neighborhoods and prove a result about their singularity locus. Then, the chapter is devoted to understand how can we build foliations of the first infinitesimal neighborhood: in Section 4.3 the main idea is to use a 2-splitting to project a foliation of the ambient manifold to a foliation of the first infinitesimal neighborhood of S, while in Section 4.4, under the hypothesis that S is splitting, I give some results about the extendability of a foliation to the first infinitesimal neighborhood.

It turns out that the properties of the atlas established in Section 4.1 give rise to the vanishing of the Atiyah class for $\mathcal{N}_{\mathcal{F},M}$. Chapter 5 is devoted to study the partial holomorphic connections arising from splittings of the Atiyah sequence. In Section 5.1 I study the partial holomorphic connection arising from the existence of a foliation of the first infinitesimal neighborhood. In Section 5.2 I study how involutive and non involutive subsheaves of the tangent sheaf to the first infinitesimal neighborhood that restricted to S are subsheaves of a foliation $\mathcal{F}$ give rise to a partial holomorphic connection on $\mathcal{N}_{\mathcal{F},M}$. Then, in Section 5.3 I give a quick survey of the results of Abate, Bracci and Tovena about the existence of partial holomorphic connections for the normal bundle N_S of a submanifold. Section 5.4 summarizes the results obtained, writing down explictly the residue theorems that stem from our treatment; then Section 5.5 is devoted to compute the variation index for our theorems for the simplest case, a dimension 1 foliation on a complex surface leaving a submanifold S invariant. Section 5.6 computes the index in the simplest transverse case. The last section of the chapter, Section 5.7, establishes an index theorem for the involutive closure of a coherent subsheaf $\mathcal{E}$ of $\mathcal{T}_S$, i.e., the smallest involutive coherent subsheaf containing $\mathcal{E}$.

Chapter 6 applies the theory to the case of holomorphic self maps; in Section 6.1 we define and prove some properties of the canonical distribution associated to a holomorphic self-map tangent to the identity. The canonical distribution is what permits us to use the results obtained for holomorphic foliations in the case of discrete dynamics. Section 6.2 is a quick survey of the results by Abate, Bracci and Tovena while Section 6.3 and 6.4 are devoted to write down explictly the index theorems we obtain and to compute some indices.

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Chapter 1

Foundational material

No material in this chapter is original; we refero to [3, 4, 5, 18, 11, 31].

1.1 Notation and conventions

In some sections (and chapters) of this thesis we use the Einstein summation convention. At the beginning of each section (or chapter) in which we use this convention we shall put a remark as follows.

Remark 1.1.1. In this section (chapter) we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

To ease the understanding of the computations the indices are going to have a fixed range. In this paper, M is a n-dimensional complex manifold, S a complex subvariety of codimension m and $\mathcal{F}$ a dimension l holomorphic foliation of either M or S, with $l \leq n-m$. Then the indices are going to have the following range:

- h, k will range in $1, \dots, n$; these are the indices relative to the coordinate system of M;

- p, q will range in $m+1, \dots, n$; in an atlas adapted to S (see definition 1.1.2); these are the indices relative to the coordinates along S;

- r, s will range in $1, \dots, m$; in an atlas adapted to S; these are the indices relative to the coordinates normal to S;

- i, j will range in $m+1, \dots, m+l$; in an atlas adapted to $\mathcal{F}$ (see definition 1.8.16); these are the indices relative to the coordinates along $\mathcal{F}$;

- u, v will range in $1, \dots, m, m+l+1, \dots, n$; in an atlas adapted to $\mathcal{F}$; these are the the indices relative to the coordinates normal to $\mathcal{F}$.

In case we shall need more indices of each type, we shall prime $'$ them or put a subscript, e.g., r_1.

We shall denote by $\mathcal{O}_M$ the structure sheaf of holomorphic functions on M, by $\mathcal{I}_S$ the ideal sheaf of a subvariety S and by $\mathcal{I}_S^k$ its k-th power as an ideal. If f

is an element of $\mathcal{O}_M$ we will denote by $[f]_{k+1}$ its image in $\mathcal{O}_{S(k)} := \mathcal{O}_M / \mathcal{I}_S^{k+1}$; we shall denote by θ_k the canonical projections from $\mathcal{O}_{S(k)}$ to $\mathcal{O}_S$

$$\theta_k : [f]_{k+1} \mapsto [f]_1$$

and by $\theta_{k,h}$, with $k > h$, the canonical projections from $\mathcal{O}_{S(k)}$ to $\mathcal{O}_{S(h)}$

$$\theta_{k,h} : [f]_{k+1} \mapsto [f]_{h+1}.$$

We denote by $\mathcal{T}_M$ and $\mathcal{T}_S$ the tangent sheaves to M and S respectively, where defined. We will denote by Ω_M the sheaf of holomorphic one forms on M and in general we write $\Omega_{M,S(k)}$ to denote the sheaf $\Omega_M \otimes \mathcal{O}_{S(k)}$.

We denote by A^p the sheaf of *smooth* p-forms on M and by $A^{p,q}$ we denote the sheaf of smooth forms of type (p, q); we will use, without any further notice the notation A^0 for the sheaf of smooth functions. In general, we will also use the notation $A^p(E)$ for the sheaf of p-forms with values in a vector bundle E.

The following is a definition we will use through the whole thesis.

Definition 1.1.2. Let $\mathcal{U}$ be an atlas for M. We say that $\mathcal{U}$ is **adapted to** S if on each coordinate neighborhood $(U_\alpha, z_\alpha^1, \ldots, z_\alpha^n)$ such that $U \cap S$ is not empty, we have that $S \cap U_\alpha = \{z_\alpha^1 = \ldots = z_\alpha^m = 0\}$, where m is the codimension of S.

1.2 Poincaré and Alexander dualities

In this section we shall present Poincaré duality and Alexander duality using the framework of simplicial homology (and cohomology); this choice was made to ease the understanding of the strong connection between those dualities and intersection theory. In the proof we assume some knowledge of simplicial homology; the line we will mainly follow is the one of the exposition in [18, Chapter 0, pag. 53]. We will use the following convention for the orientation of the boundary of a manifold M: let $p \in \partial M$, if $v_1, v_2, \ldots, v_n$ is a positively orientated basis for T_pM, we say $v_1, \ldots, v_{n-1}$ is a positively oriented basis for ∂M if v_n is pointing outward ∂M (this is not coherent with the orientation convention we use later on but makes the proofs of this section clearer). Suppose now M is a manifold of dimension n and $K = (\sigma_\alpha^k, \partial)_{\alpha,k}$ is a triangulation of M, where k denotes the dimension of the simplex. First of all we take the first barycentric subdivision of K, the simplicial complex $K' = (\tau_\alpha^k, \partial)_{\alpha,k}$. Now, for each vertex σ_α^0 we denote by $*\sigma_\alpha^0$ the union of the faces of K' having σ_α^0 as a vertex; formally:

$$*\sigma_\alpha^0 := \bigcup_{\sigma_\alpha^0 \in \tau_\beta^n} \tau_\beta^n.$$

Now, we define $*\sigma_\alpha^k$ for each simplex in the original subdivision:

$$\Delta_\alpha^{n-k} := *\sigma_\alpha^k := \bigcap_{\sigma_\beta^0 \in \sigma_\alpha^k} *\sigma_\beta^0.$$

Taking the cells $\{\Delta_\alpha^{n-k}\}$ we have a new decomposition of M, called the **dual cell decomposition with respect to** K; we claim there exists a a coboundary operator δ for this decomposition with a good behaviour with respect to the intersection of cycles and which permits us to construct an isomorphism between this "dual complex" and the cohomology of the complex.

We fix some notation: fixed a σ_α^k we take all τ_β^n such that $\sigma_\alpha^k \cap \tau_\beta^n \neq \emptyset$; their intersection coincides with the intersection $*\sigma_\alpha^k \cap \sigma_\alpha^k$ and it is the barycenter of σ_α^k, that, from now on, we will denote by p_α^k. Moreover $*\sigma_\alpha^k$ is the only cell of the dual cell decomposition meeting σ_α^k, intersecting it transversely.

Definition 1.2.1. Let M be an oriented n-manifold, let A and B be two piecewise smooth cycles on M of dimension k and $n-k$ respectively and suppose p is a point of transverse intersection of A and B. Let $v_1, \ldots, v_k$ be an oriented basis of T_pA and $w_{k+1}, \ldots, w_n$ an oriented basis for T_pB; the **intersection index** $\iota_p(A \cdot B)$ of A with B at p is $+1$ if $v_1, \ldots, v_k, w_{k+1}, \ldots, w_n$ is an oriented basis basis for T_pM, and -1 otherwise. If A and B intersect transversely everywhere, we define the intersection number $(A \cdot B)$ to be

$$(A \cdot B) = \sum_{p \in A \cap B} \iota_p(A \cdot B).$$

We can put an orientation on $\{\Delta_\alpha^{n-k}\}$ such that, in p_α^k, the standard oriented basis of the tangent space to σ_α^k followed by such a basis on $\{\Delta_\alpha^{n-k}\}$ is positively oriented (their intersection index $\iota_{p_\alpha^k}(\sigma_\alpha^k, \Delta_\alpha^{n-k}) = +1$). We denote now by σ_α^j one of the faces of the boundary of σ_α^k and let $\Delta_\alpha^j := *\sigma_\alpha^j$, these two simplexes meet in the barycenter that we will denote by p_α^j. We want to compute the index $\iota_{p_\alpha^j}(\sigma_\alpha^j, \Delta_\alpha^j)$.

The intersection $\sigma_\alpha^k \cap \Delta_\alpha^j$ is a path between p_α^k and p_α^j; we parametrize it with a differentiable curve $\gamma : [0,1] \to \sigma_\alpha^k$ and we take vector fields

$$v_1(t), \ldots, v_{k-1}(t), w_{k+1}(t), \ldots, w_n(t)$$

along γ such that

$$v_1(t), \ldots, v_{k-1}(t)$$

are tangent to σ_α^k and

$$w_{k+1}(t), \ldots, w_n(t)$$

are tangent to Δ_α^{n-k} and

$$v_1(t), \ldots, v_{k-1}(t), \gamma'(t), w_{k+1}(t), \ldots, w_n(t)$$

is a frame for $T_{\gamma(t)}M$ for every $t \in [0,1]$. Moreover we require that

$$v_1(0), \ldots, v_{k-1}(0), \gamma'(0)$$

is a positive basis of $T_{\gamma(0)}\sigma_\alpha^k$ and

$$w_{k+1}(0), \ldots, w_n(0)$$

is a basis of $T_{\gamma(0)}\Delta_\alpha^{n-k}$ such that

$$v_1(0), \ldots, v_{k-1}(0), \gamma'(0), w_{k+1}(0), \ldots, w_n(0)$$

is a positively oriented basis for $T_{\gamma(0)}M$.

We follow the path from p^k_α to p^j_α; in p^j_α the frame $v_1(1), \dots, v_{k-1}(1)$ is positively oriented for σ^j_α, since $\gamma'(1)$ is pointing outward. The basis

$$\gamma'(1), w_{k+1}(1), \dots, w_n(1)$$

has sign $(-1)^{n-k-1}$ with respect to the basis

$$w_{k+1}(1), \dots, w_n(1), \gamma'(1)$$

and this basis has negative sign with respect with the orientation of Δ^{n-k}_α since $\gamma'(1)$ is pointing inward. This implies that $\iota_{p^j_\alpha}(\sigma^j_\alpha, \Delta^j_\alpha) = (-1)^{n-k}$.

We define the coboundary operator δ for the dual cell decomposition to be

$$\delta(\Delta^{n-k}_\alpha) = (-1)^{n-k} * (\partial \sigma^k_\alpha).$$

Using the dual simplicial complex $K^* = (\Delta^{n-k}_\alpha, \delta)$ we can prove now the duality theorems.

Theorem 1.2.2 (Poincaré duality)**.** *Let M be a compact, oriented, manifold of dimension n; then the intersection pairing*

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$$

is unimodular; i.e., any linear functional $T : H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$ is expressible as intersection with some class $\alpha \in H_k(M, \mathbb{Z})$, and any class $\alpha \in H_k(M, \mathbb{Z})$ having intersection number 0 with all classes in $H_{n-k}(M, \mathbb{Z})$ is a torsion class.

Proof. First of all we remark that the map P which sends σ^k_α in Δ^{n-k}_α is an isomorphism between the complex K of the original cell decomposition and the complex K^* of cochains in the dual cell decomposition. This map induces an isomorphism

$$P_M : H_k(M, \mathbb{Z}) \to H^{n-k}(M; \mathbb{Z})$$

such that

$$(\gamma \cdot \sigma) = P(\gamma)(\sigma)$$

for γ in $H_k(M, \mathbb{Z})$ and σ in $H_{n-k}(M, \mathbb{Z})$. □

The same way of reasoning we proves another important theorem, which gives us a duality theorem for relative homology. Suppose now we have a submanifold S in M and that we take the relative homology of the pair $(M, M \setminus S)$; now, this means that we are taking into account only the cycles in the homology of M which are contained in the subcomplex of K defined by S. Please note that a simplex σ of K is contained in S if and only if its dual cell $*\sigma$ intersects with S.

Theorem 1.2.3 (Alexander duality)**.** *Let S be a compact submanifold of M, oriented manifold of dimension n; the intersection pairing*

$$H_k(M, M \setminus S; \mathbb{Z}) \times H_{n-k}(S; \mathbb{Z}) \to \mathbb{Z},$$

is unimodular; i.e., any linear functional $\tau : H^{n-k}(S, \mathbb{Z}) \to \mathbb{Z}$ is expressible as intersection with some class $\alpha \in H_k(M, M \setminus S, \mathbb{Z})$, and any class $\alpha \in H_{n-k}(S, \mathbb{Z})$ having intersection number 0 with all classes in $H_k(M, M \setminus S; \mathbb{Z})$ is a torsion class.

1.3 Čech Cohomology

Definition 1.3.1. Let X be a topological space. A **presheaf** $\mathcal{G}$ of groups (abelian groups, rings, vector spaces) over X is a correspondance which associates to every open set U of X a group (abelian group, ring, vector space) $\mathcal{G}(U)$ and to each pair of nested subsets $U \subset V$ a homomorphism of groups (abelian groups, rings, vector spaces) $r_{U,V} : \mathcal{G}(V) \to \mathcal{G}(U)$ such that:

1. $\mathcal{G}(\emptyset) = 0$;
2. $r_{U,U} = \mathrm{id}_{\mathcal{G}(U)}$;
3. for each triple $U \subset V \subset W$ we have that $r_{U,V} \circ r_{V,W} = r_{U,W}$.

We say a presheaf $\mathcal{G}$ is a **sheaf** if for every $U = \bigcup U_i$ union of open sets of X the following additional conditions are satisfied:

1. if $f, g \in \mathcal{G}(U)$ are such that $r_{U_i,U}(f) = r_{U_i,U}(g)$ for every i, then $f \equiv g$;
2. if f_i are a collection of elements of $\mathcal{G}(U_i)$ such that
$$r_{U_i \cap U_j, U_i}(f_i) = r_{U_i \cap U_j, U_j}(f_j),$$
then there exists an $f \in \mathcal{G}(U)$ such that $r_{U_i,U}(f) = f_i$.

Definition 1.3.2. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a topological space X. A **morphism of sheaves** α is a collection of homomorphisms $\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \subset X}$ such that, if $U \subset V \subset X$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\alpha_V"] \arrow[d, "r_{U,V}"] & \mathcal{G}(V) \arrow[d, "r_{U,V}"] \\ \mathcal{F}(U) \arrow[r, "\alpha_U"] & \mathcal{G}(U). \end{tikzcd}$$

The **kernel** of α is the sheaf $\ker(\alpha)$ given by

$$\ker(\alpha)(U) = \ker(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

Remark 1.3.3. The definition of the **cokernel** of α is more difficult. We define a section of $\mathrm{coker}(\alpha)$ over an open set U to be a cover $\{U_\alpha\}$ of U and sections $\sigma_\alpha \in \mathcal{G}(U_\alpha)$ such that, for every pair α, β

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in \alpha(\mathcal{F}(U_\alpha \cap U_\beta));$$

we define an equivalence relation on these collections saying that $\{(U_\alpha, \sigma_\alpha)\}$ is equivalent to $\{(U'_\alpha, \sigma'_\alpha)\}$ if for every $p \in U$, $p \in U_\alpha \cap U'_\beta$ there exists a neighborhood $V \subset U_\alpha \cap U'_\beta$ of p such that $\sigma_\alpha|_V - \sigma'_\beta|_V \in \alpha_V(\mathcal{F}(V))$.

Definition 1.3.4. We say a sequence of sheaf maps

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

is **exact** if $\mathcal{E} = \ker(\alpha)$ and $\mathcal{G} = \mathrm{coker}(\beta)$.

Definition 1.3.5. Let $\mathcal{R}$ be a sheaf of rings on a topological space X with restriction maps $r_{U,V}$. Let $\mathcal{M}$ be a sheaf on X such that $\mathcal{M}(U)$ is a $\mathcal{R}(U)$ module for each U, with restriction maps $s_{U,V}$. We say that $\mathcal{M}$ is a **sheaf of $\mathcal{R}$-modules** if, for $U \subset V$, $t \in \mathcal{R}(U)$, $m \in \mathcal{M}(U)$ we have that

$$s_{U,V}(t \cdot m) = r_{U,V}(t) \cdot s_{U,V}(m).$$

Remark 1.3.6. In this thesis we use many different sheaves. In Section 1.1 we already cited the sheaf of holomorphic functions on a complex manifold M, denoted by $\mathcal{O}_M$, which is a sheaf of rings and the sheaf of ideals of a subvariety S, denoted by $\mathcal{I}_S$.

Examples of sheaf of $\mathcal{O}_M$-modules are the tangent sheaf of M, denoted by $\mathcal{T}_M$ and the sheaf of holomorphic one forms on M, denoted by Ω_M.

The sheaf of smooth functions on M, denoted by C^∞ is also a sheaf of ring, while the sheaf of smooth p-forms on M, denoted by A^p is a sheaf of C^∞-modules.

Suppose now M is a smooth manifold of dimension m and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of M. For a $(p+1)$-uple $(\alpha_0, \dots, \alpha_p)$ of elements of I, we set $U_{\alpha_0 \dots \alpha_p} = \cap_{\nu=0}^p U_{\alpha_\nu}$. We define the p-**cochains** of the Čech complex of the sheaf $\mathcal{G}$ to be the elements of:

$$C^p(\mathcal{U}, \mathcal{G}) = \prod_{(\alpha_0 \dots \alpha_p) \in I^{p+1}} \mathcal{G}(U_{\alpha_0 \dots \alpha_p}).$$

We define the **coboundary operator** $\delta : C^p(\mathcal{U}, \mathcal{G}) \to C^{p+1}(\mathcal{U}, \mathcal{G})$ by:

$$(\delta\sigma)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{\nu=0}^{p+1} (-1)^\nu \sigma_{\alpha_0 \dots \hat{\alpha}_\nu \dots, \alpha_{p+1}},$$

where the hat means that the index is to be omitted. We will call a p-cochain α a **cocycle** if its image $\delta(\alpha)$ is 0, and we will call a p-cochain α a **coboundary** if $\alpha = \delta(\beta)$ for some $(p-1)$-cochain β.

Lemma 1.3.7. *Let δ be the coboundary operator of Čech cohomology; then $\delta \circ \delta = 0$*

Proof. The proof is a computation:

$$\begin{aligned}(\delta^2\sigma)_{\alpha_0, \dots, \alpha_{p+2}} &= \sum_{\nu=0}^{p+2} (-1)^\nu \delta\sigma_{\alpha_0 \dots \hat{\alpha}_\nu \dots \alpha_{p+2}} \\ &= \sum_{\nu=0}^{p+2} (-1)^\nu \sum_{\mu=0}^{p+1} (-1)^\mu \sigma_{\alpha_0 \dots \hat{\alpha}_\nu \dots \hat{\alpha}_\mu \dots \alpha_{p+1}} = 0,\end{aligned}$$

since every component appears twice in this sum, with opposite signs. □

Since $\delta \circ \delta = 0$ we have a complex:

$$0 \to C^0(\mathcal{U}; \mathcal{G}) \to C^1(\mathcal{U}; \mathcal{G}) \to C^2(\mathcal{U}; \mathcal{G}) \to C^3(\mathcal{U}; \mathcal{G}) \to \cdots$$

which we denote by $C^*(\mathcal{U};\mathcal{G},\delta)$ or $C^*(\mathcal{U};\mathcal{G})$ and is called the **Čech complex** for the sheaf $\mathcal{G}$. We denote by $H^p(\mathcal{U},\mathcal{G})$ its p-th cohomology group

$$H^p(\mathcal{U},\mathcal{G}) := \frac{\ker(\delta^p : C^p(\mathcal{U};\mathcal{G}) \to C^{p+1}(\mathcal{U};\mathcal{G}))}{\operatorname{im}(\delta^{p-1} : C^{p-1}(\mathcal{U};\mathcal{G}) \to C^p(\mathcal{U};\mathcal{G}))}.$$

Since $\ker\delta^0 = \mathcal{G}(M)$ we have $H^0(\mathcal{U},\mathcal{G}) = \mathcal{G}(M)$, the global sections of $\mathcal{G}$; we shall also denote $\mathcal{G}(M)$ by $\Gamma(M,\mathcal{G})$. The **Čech cohomology for the sheaf** $\mathcal{G}$ denoted by $\check{H}^p(M,\mathcal{G})$ is defined by taking the direct limit of $H^p(\mathcal{U},\mathcal{G})$ over the open coverings $\mathcal{U}$ of M. In practice we are going to work with covers, thanks to the Leray theorem, that we state here.

Theorem 1.3.8 (Leray theorem)**.** *If the covering $\mathcal{U}$ is acyclic for the sheaf $\mathcal{G}$ in the sense that*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{G}) = 0, \quad q > 0, \text{ any } i_1, \dots, i_p,$$

then $H^*(\mathcal{U},\mathcal{G}) \cong \check{H}^*(M,\mathcal{G})$.

Theorem 1.3.9 (Long exact sequence for Čech cohomology)**.** *Suppose that*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0,$$

is a short exact sequence of sheaves. Then, there exists operators $\delta^* : H^p(M,\mathcal{G}) \to H^{p+1}(M,\mathcal{E})$ such that the following sequence is exact:

$$\cdots \longrightarrow H^p(M,\mathcal{E}) \xrightarrow{\alpha^*} H^p(M,\mathcal{F}) \xrightarrow{\beta^*} H^p(M,\mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(M,\mathcal{E}) \longrightarrow \cdots$$

Proof. Let $[\sigma]$ be a class in $H^p(M,\mathcal{G})$ and let $(\mathcal{U},\sigma)$ be a Čech cocycle representing it. Passing to a refinement $\mathcal{U}'$ of $\mathcal{U}$ we can find $\tau \in C^p(\mathcal{U}',\mathcal{F})$ such that $\beta(\tau) = \sigma$. We take $\delta(\tau) \in C^{p+1}(\mathcal{U}',\mathcal{F})$. We have that $\beta(\delta(\tau)) = \delta(\beta(\tau)) = \delta\sigma = 0$. So, taking a refinement $\mathcal{U}''$ of $\mathcal{U}'$ we have that there exists $\mu \in C^{p+1}(\mathcal{U}'',\mathcal{E})$ such that $\alpha(\mu) = \delta(\tau)$. Now

$$\alpha\delta\mu = \delta\alpha\mu = \delta^2\tau = 0.$$

Since α is injective we have that $\delta\mu = 0$ and therefore $[\mu]$ is a well defined class in $H^{p+1}(M,\mathcal{E})$. We define $\delta^*([\sigma]) = [\mu]$. □

We will now prove an acyclicity result for the sheaf of smooth q-forms.

Lemma 1.3.10. *Let M be a smooth manifold and let $\mathcal{U}$ be a cover of M, locally finite. Then the Čech complex of q-forms on $\mathcal{U}$ is acyclic, i.e.*

$$H^p(\mathcal{U};A^q) = 0,$$

for $p > 0$.

Proof. Since M is a manifold and $\mathcal{U}$ is locally finite there exists a partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}$, which we will denote by ρ_α. We construct now an operator

$$K : C^p(\mathcal{U},A^q) \to C^{p-1}(\mathcal{U},A^q)$$

in the following way: if σ is an element of $C^p(\mathcal{U}, A^q)$ then

$$(K\sigma)_{\alpha_0\dots\alpha_{p-1}} = \sum_\alpha \rho_\alpha \sigma_{\alpha\alpha_0\dots\alpha_{p-1}}.$$

Suppose now σ is a cocycle. Then:

$$(\delta\sigma)_{\alpha\alpha_0\dots\alpha_p} = \sigma_{\alpha_0\dots\alpha_p} + \sum_i (-1)^{i+1} \sigma_{\alpha\alpha_0\dots\hat{\alpha_i}\dots\alpha_p} = 0;$$

and therefore

$$\sigma_{\alpha_0\dots\alpha_p} = \sum_i (-1)^i \sigma_{\alpha\alpha_0\dots\hat{\alpha_i}\dots\alpha_p}.$$

If $\tau = K\sigma$ then

$$\begin{aligned}(\delta\tau)_{\alpha_0\dots\alpha_p} &= \sum_i (-1)^i \tau_{\alpha_0\dots\hat{\alpha_i}\dots\alpha_p} \\ &= \sum_i (-1)^i \sum_\alpha \rho_\alpha \sigma_{\alpha\alpha_0\dots\hat{\alpha_i}\dots\alpha_p} = \sum_\alpha \sum_i (-1)^i \rho_\alpha \sigma_{\alpha\alpha_0\dots\hat{\alpha_i}\dots\alpha_p} \\ &= \sum_\alpha \rho_\alpha \sigma_{\alpha_0\dots\alpha_p} = \sigma_{\alpha_0\dots\alpha_p}.\end{aligned}$$

This means that every cocycle is a coboundary. □

Remark 1.3.11. This result and Leray Theorem 1.3.8 imply that the Čech cohomology groups $\check{H}^p(M, A^q) = 0$ for $p > 0$.

Remark 1.3.12. An important class of sheaves are the so called **fine sheaves**. A fine sheaf $\mathcal{F}$ is a sheaf that for any open cover $\{U_i\}$ of a topological space X admits a locally finite family of homomorphisms $\alpha_{U_i} : \mathcal{F} \to \mathcal{F}$ such that for every x in X we have that

$$\sum_{x \in U_i} \alpha_{U_i} = \mathrm{id}$$

and $\alpha_{U_i} \equiv 0$ outside U_i. For such sheaves the proof of Lemma 1.3.10 works word by word. When dealing with holomorphic objects one of the big issues is that sheaves as the sheaf of holomorphic functions and holomorphic forms do **not** admit a partition of unity.

1.4 Čech-de Rham cohomology

We will now define a new chain complex, the Čech-de Rham complex, which is going to be the main tool in our study of Chern classes. We refer to [31, 11] for the material in this section. Let M be a m dimensional smooth manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. We take the cochain groups

$$K^{p,q} := C^p(\mathcal{U}, A^q) = \prod_{\alpha_0,\dots,\alpha_p} A^q(U_{\alpha_0\dots\alpha_p}).$$

We have two coboundary operators:

$$\delta : K^{p,q} \to K^{p+1,q},$$

the Čech coboundary defined in the previous section and

$$d : K^{p,q} \to K^{p,q+1},$$

the exterior differential on each U_α. We define now a new cochain complex, denoted by $A^\bullet(\mathcal{U})$; the cochain groups are given by $A^r(\mathcal{U}) = \bigoplus_{p+q=r} K^{p,q}$ and the differential $D = D^r : A^r(\mathcal{U}) \to A^{r+1}(\mathcal{U})$ is given by

$$(D\sigma)_{\alpha_0 \dots \alpha_p} = \sum_{\nu=0}^{p} (-1)^\nu \sigma_{\alpha_0 \dots \hat{\alpha}_\nu \dots \alpha_p} + (-1)^p d\sigma_{\alpha_0 \dots \alpha_p}.$$

We could write D in a compact form saying that $D = \delta + (-1)^p d$. With this form in mind it is simple to prove that $D^2 = 0$, since

$$D^2 = D(\delta + (-1)^p d) = \delta^2 + (-1)^{p+1} d\delta + (-1)^p \delta d + (-1)^p d^2 = 0,$$

because d and δ commute.

Theorem 1.4.1. *The restriction map $r^* : A^r(M) \to C^0(\mathcal{U}, A^r)$ induces an isomorphism*

$$H^r_{dR}(M; \mathbb{C}) \to H^r(A^\bullet(\mathcal{U})).$$

Proof. First of all, we prove that each class $[\omega]$ in $H^r(A^\bullet(\mathcal{U}))$ can be represented by a cocycle with only the top component, i.e., the component in $K^{0,r}$. We take any cocycle representing $[\omega]$, denoting it by ω. Suppose its lowest nonzero component, denoted by α, lies is in $K^{k,r-k}$. Since ω is a cocycle, we have that $D\omega = 0$ and in particular $\delta\alpha = 0$, since α is the lowest nonzero component. Thanks to Remark 1.3.11 we know that there exists $\beta \in K^{k-1,r-k}$ such that $\beta = \delta\alpha$; let now $\omega' = \omega - D\beta$. We have that $[\omega'] = [\omega]$ but ω' has no component in dimension lower than $K^{k-1,r-k+1}$; iterating this process we obtain a cocycle which has only top component. Being a cocycle for an element $\sigma \in K^{0,r}$ means that $\delta\sigma = 0$ and $d\sigma = 0$; this means that σ is a closed global r-form. Therefore, the restriction map is surjective because each class of $H^r(A^\bullet(\mathcal{U}))$ can be represented by a closed global form.

We prove now that it is injective. Suppose ω is a class in $H^r_{dR}(M; \mathbb{C})$ such that $r^*(\omega) = D\phi$ for ϕ a cochain. As we proved above, we can find a cocycle η homologous to $D\phi$ that has only top component. Now, we can find a cochain θ such that $D\theta = \eta$; since η has only top component, if we denote by β the lowest component of θ we have that $\delta\beta = 0$. Using the same procedure as before we can find a new cochain θ' such that the difference $\theta - \theta'$ is exact and which has only top component. This implies $D\theta' = d\theta' = \eta$ and that $\delta\theta' = 0$; therefore θ' is a global form whose differential is ω. □

To compute the cohomology of the costant sheaf $\mathbb{C}$ we use an important result, called the Poincaré Lemma. We refer, for the proof to [11, pag. 33].

Lemma 1.4.2 (Poincaré Lemma)**.** *Let U be a simply connected open set. Then $H^p(U, \mathbb{C}) = 0$ for $p > 0$.*

From the Poincaré Lemma follows directly the acyclicity of the constant sheaf $\mathbb{C}$, and the Leray theorem yields.

Lemma 1.4.3. *Let $\mathcal{U}$ be a good cover, i.e., every non-empty finite intersection $U_{\alpha_0\ldots\alpha_p}$ is simply connected. Then*

$$H^r(A^\bullet(\mathcal{U})) \simeq \check{H}^r(M;\mathbb{C})$$

Corollary 1.4.4. *Suppose M admits a locally finite good cover $\mathcal{U}$. Then*

$$H^r_{dR}(M;\mathbb{C}) \simeq \check{H}^r(M;\mathbb{C}).$$

All the usual properties of de Rham cohomology pass, thanks to the isomorphism, to Čech-de Rham cohomology. The cup product in Čech-de Rham cohomology is the analogous of the wedge product in de Rham cohomology (and on forms):

$$\smile : A^r(\mathcal{U}) \times A^s(\mathcal{U}) \to A^{r+s}(\mathcal{U}).$$

The cup product between two classes $\sigma \in A^r(\mathcal{U})$ and $\tau \in A^s(\mathcal{U})$ is given in components by:

$$(\sigma \smile \tau)_{\alpha_0\ldots\alpha_p} = \sum_{\nu=0}^{p} (-1)^{r-\nu}(-1)^{p-\nu}\sigma_{\alpha_0\ldots\alpha_\nu} \wedge \tau_{\alpha_\nu\ldots\alpha_p}.$$

Let M be an oriented manifold of dimension n and let S be a closed set. We define now the isomorphism between the relative cohomology $H^*(M, M\setminus S, \mathbb{C})$ and a cohomology $H^*(A^\bullet(\mathcal{U}, U_0))$, that we will define now. Let $U_0 = M \setminus S$, $U_1 = M$ and let $\mathcal{U} = \{U_0, U_1\}$. We denote by $A^r(\mathcal{U}, U_0)$ the kernel of the canonical projection $A^r(\mathcal{U}) \to A^r(U_0)$. We have the short exact sequence of cochain groups:

$$0 \to A^r(\mathcal{U}, U_0) \to A^r(\mathcal{U}) \to A^r(U_0) \to 0,$$

which gives rise to the long exact sequence:

$$\cdots \to H^{r-1}(A^\bullet(U_0)) \to H^r(A^\bullet(\mathcal{U}, U_0)) \to H^r(A^\bullet(\mathcal{U})) \to H^r(A^\bullet(U_0)) \to \cdots$$

From the isomorphism between Čech-de Rham cohomology and de Rham cohomology we know that $H^*(A^\bullet(U_0)) \simeq H^*(M\setminus S, \mathbb{C})$ and $H^*(A^\bullet(\mathcal{U})) \simeq H^*(M, \mathbb{C})$ so, comparing with the long exact sequence in cohomology for the pair $(M, M\setminus S)$,

$$\cdots \to H^{r-1}(M\setminus S;\mathbb{C}) \to H^r(M, M\setminus S;\mathbb{C}) \to H^r(M) \to H^r(M\setminus S;\mathbb{C}) \to \cdots$$

we have, by the five lemma that $H^r(A^\bullet(\mathcal{U}, U_0)) \simeq H^r(M, M\setminus S;\mathbb{C})$. We study now the behaviour of the cup product of $A^k(\mathcal{U}) \times A^l(\mathcal{U}) \to A^{k+l}(\mathcal{U})$ given by

$$(\sigma_0, \sigma_1, \sigma_{01}) \smile (\tau_0, \tau_1, \tau_{01}) = (\sigma_0 \wedge \tau_0, \sigma_1 \wedge \tau_1, (-1)^k \sigma_0 \wedge \tau_{01} + \sigma_{01} \wedge \tau_1).$$

If σ is in $A^\bullet(\mathcal{U}, U_0)$ then $\sigma_0 = 0$ and the cup product

$$(0, \sigma_1, \sigma_{01}) \smile (\tau_0, \tau_1, \tau_{01}) = (0, \sigma_1 \wedge \tau_1, \sigma_{01} \wedge \tau_1) \tag{1.1}$$

depends only on $\sigma_1, \sigma_{01}, \tau_1$:

$$\smile : A^k(\mathcal{U}, U_0) \times A^l(U_1) \to A^{k+l}(\mathcal{U}, U_0).$$

For any open neighborhood U of S in M, thanks to the excision principle, the inclusion of pairs $(U, U \setminus S) \hookrightarrow (M, M \setminus S)$ induces an isomorphism

$$H^r(M, M \setminus S; \mathbb{C}) \xrightarrow{\simeq} H^r(U, U \setminus S; \mathbb{C}).$$

If U is a tubular neighborhood of S we can take now another cover of M, given by $U_0 = M \setminus S$ and $U_1 = U$; thanks to the excision principle we see that the cup product above induces a cup product

$$H^k(M, M \setminus S; \mathbb{C}) \times H^l(S; \mathbb{C}) \to H^{k+l}(M, M \setminus S; \mathbb{C}).$$

1.5 Integration in Čech-de Rham cohomology

In this section we will show how integration is defined in the Čech-de Rham formalism (we refer to [31, 24]). Let M be a n dimensional smooth manifold and let $\mathcal{U}$ be an open cover of M indexed as in Section 1.3 and 1.4. Given an open subset V of M, we denote by $\operatorname{Int} V$ the interior of V. Given a p-uple of indices $\{\alpha_0, \dots, \alpha_p\}$ we say it is maximal if $U_{\alpha\alpha_0\dots\alpha_p} \neq \emptyset$ implies that $\alpha \in \{\alpha_0, \dots, \alpha_p\}$.

Definition 1.5.1. A **system of honeycomb cells** adapted to $\mathcal{U}$ is a collection of open subsets $\{R_\alpha\}$ with piecewise smooth boundary in M satisfying the following conditions:

1. $R_\alpha \subset U_\alpha$,
2. $M = \bigcup_\alpha R_\alpha$,
3. $\operatorname{Int} R_\alpha \cap \operatorname{Int} R_\beta = \emptyset$ if $\alpha \neq \beta$,
4. if $U_{\alpha_0\dots\alpha_p} \neq \emptyset$, then $R_{\alpha_0\dots\alpha_p} = \bigcap_{\nu=0}^{p} R_{\alpha_\nu}$ is an $(n-p)$-manifold with piecewise smooth boundary
5. if $\{\alpha_0, \dots, \alpha_p\}$ is maximal, $R_{\alpha_0\dots\alpha_p}$ has no boundary.

If M is oriented we orient $R_{\alpha_0\dots\alpha_p}$ by the following convention:

1. each R_α has the same orientation as M and the boundary has the **boundary orientation**, i.e., if p is in ∂R_α and w is an outward pointing vector in $T_p R_\alpha$ we say $v_1, \dots, v_{n-1}$ is a positively oriented basis of $T_p \partial R_\alpha$ if $w, v_1, \dots, v_{n-1}$ is a positively oriented basis of $T_p R_\alpha$. In general, given a $R_{\alpha_0\dots\alpha_p}$ its boundary is oriented following the same convention;
2. if ρ is a permutation, the orientation of $R_{\alpha_{\rho(0)},\dots,\alpha_{\rho(p)}} = \operatorname{sign}\rho \cdot R_{\alpha_0\dots\alpha_p}$,
3. $\partial R_{\alpha_0\dots\alpha_p} = \sum_\alpha R_{\alpha_0\dots\alpha_p\alpha}$.

A complete reference about the existence and the use of honeycomb systems can be found in [24].

Let M be an oriented compact n dimensional manifold, if we have an open covering $\mathcal{U}$ and a system of honeycomb cells we can now define the integration

$$\int_M : A^n(\mathcal{U}) \to \mathbb{C}$$

by
$$\int_M \sigma = \sum_{p=0}^{n} \sum_{\alpha_0\ldots\alpha_p} \int_{R_{\alpha_0\cdots\alpha_p}} \sigma_{\alpha_0\ldots\alpha_p}.$$

If σ is a Čech-de Rham n-cocycle, we can represent it as a cochain with only top coefficients which is the image of the restriction of a closed global n-form. Therefore, using a partition of unity argument, it can be seen that this integration coincides with the usual integration of a de Rham n-class [11]. So, the integral of a n-cocycle does not depend on the honeycomb system we take into account and, if a n-cochain is a coboundary, its integral is 0. So we have a well defined integration on the cohomology $H^n(A^\bullet(\mathcal{U}))$. We can now translate Poincaré duality in the language of Čech-de Rham cohomology: the bilinear pairing
$$H^k(A^\bullet(\mathcal{U})) \times H^{n-k}(A^\bullet(\mathcal{U})) \to \mathbb{C}$$
$$(\sigma, \tau) \mapsto \int_M \sigma \smile \tau$$
gives rise to an isomorphism that sends a class $[\sigma]$ in $H^k(A^\bullet(\mathcal{U}))$ to the class $[C]$ in $H_{n-k}(M, \mathbb{C})$ such that:
$$\int_M \sigma \smile \tau = \int_C \tau,$$
for each τ in $H^{n-k}(A^\bullet(\mathcal{U}))$ where we choose the cycle C in its homology class so that it is transverse to each $R_{\alpha_0\ldots\alpha_p}$, and the integral on the right hand side is defined as
$$\sum_{p=0}^{n} \sum_{\alpha_0\ldots\alpha_p} \int_{R_{\alpha_0\cdots\alpha_p}\cap C} \tau_{\alpha_0\ldots\alpha_p}.$$

Suppose now we have an oriented manifold M of dimension n, and a compact subset S (M does not need to be compact). We take the cover $\mathcal{U} = \{U_0, U_1\}$, where $U_0 = M \setminus S$ and U_1 is a tubular neighborhood of S. We can define the integration
$$\int_M : A^n(\mathcal{U}, U_0) \to \mathbb{C}$$
by setting
$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}, \tag{1.2}$$
which induces an integration on the cohomology
$$\int_M : H^n(A^\bullet(\mathcal{U}, U_0)) \to \mathbb{C}.$$

Thanks to the cup product for the relative cohomology (1.1) we can define a pairing
$$H^k(A^\bullet(\mathcal{U}, U_0), \mathbb{C}) \times H^{n-k}(A^\bullet(U_1), \mathbb{C})$$
$$(\sigma, \tau) \mapsto \int_M \sigma \smile \tau.$$

The class $[\sigma]$ in $H^k(A^\bullet(\mathcal{U}, U_0))$ corresponds to the class $[C]$ in $H_{n-k}(S;\mathbb{C})$ such that

$$\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 = \int_C \tau_1.$$

Another interesting operation connected with integration of cochains in Čech-de Rham cohomology is the analogous of **integration along the fiber**. We recall now the definition of integration along a fiber for forms on a fiber bundle. Let $\pi : B \to M$ be a fiber bundle with typical fiber F; if M is a manifold without boundary but the fiber F has boundary ∂F then the fiber bundle B is a manifold with boundary ∂B, a fiber bundle with typical fiber ∂F; if we denote by $\partial\pi = \pi|_{\partial B}$ then $\partial\pi : \partial B \to M$ denotes the induced fiber bundle with fiber ∂F.

Let M be a smooth manifold without boundary and F is a compact oriented manifold of dimension r with boundary ∂F. Suppose B is an oriented smooth fiber bundle with fiber F. Let $\{U_\alpha\}$ be a cover of M such that there exist trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ oriented accordingly to the orientation of B. By taking smaller U_α we may assume that each U_α is a coordinate neighborhood with coordinates $(x^1_\alpha, \dots, x^n_\alpha)$. If F has boundary, we think of F as being inside an oriented manifold F' of the same dimension without boundary; we may think of B as being inside a fiber bundle $\pi' : B' \to M$ with fiber F' and each ϕ_α as being the restriction of a trivialization $\phi'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F'$ of B'. We cover F' by coordinate neighborhoods V_λ, such that $(t^1_\lambda, \dots, t^r_\lambda)$ is a coordinate system oriented accordingly to the orientation of F'; moreover, we suppose the coordinate system is such that if $\partial F \cap V_\lambda \neq \emptyset$ we have that $F \cap V_\lambda = \{t^r_\lambda \geq 0\}$; we give ∂F the boundary orientation, i.e. ∂F is oriented so that the form $(-1)^r dt^1_\lambda \wedge \dots \wedge dt^{r-1}_\lambda$ is positive. We take a covering of B given by $\mathcal{W} = \{W_{\alpha,\lambda}\}$, where $W_{\alpha,\lambda} = (\phi'_\alpha)^{-1}(U_\alpha \times V_\lambda)$. Take now a p-form ω on a neighborhood of B; the restriction morphisms to each $W_{\alpha,\lambda}$ give rise to a Čech cocycle in $C^0(A^p, \mathcal{W})$. On each $W_{\alpha,\lambda}$ the form $\omega_{\alpha,\lambda} := \omega|_{W_{\alpha,\lambda}}$ can be written as

$$\omega_{\alpha,\lambda} = \sum_{\#I+\#J=p, \#J\neq r} f_{\alpha,\lambda,I,J} dt^J_\lambda \wedge dx^I_\alpha + \sum_{\#I=p-r} f_{\alpha,\lambda,r,I} dt^\lambda \wedge dx^I_\alpha,$$

where I and J are $\#I$-tuples and $\#J$-tuples respectively of indices and by dt^λ we mean the r-form $dt^1_\lambda \wedge \dots \wedge dt^r_\lambda$. Since B is a fiber bundle we have that $(x_\beta, t_\mu) = (x_\beta(x_\alpha), t_\mu(x_\alpha, t_\lambda))$. This implies that under changes of coordinates, we have that:

$$\sum_{\#I=p-r} f_{\beta,\mu,r,I} dt^\mu dx^I_\beta = \sum_{\#I=p-r} f_{\alpha,\lambda,r,I} \det\left(\frac{\partial t_\mu}{\partial t_\lambda}\right) dt^\lambda \frac{\partial x^I_\alpha}{\partial x^J_\beta} dx^J_\beta. \tag{1.3}$$

In particular, if we look at the coordinate changes between $W_{\alpha,\mu}$ and $W_{\alpha,\lambda}$ we see that the collection of forms

$$\tilde{\omega}_{\alpha,\lambda,I} := f_{\alpha,\lambda,r,I} dt^r_\lambda$$

give rise to a well defined r-form on the fiber, that we shall denote by $\tilde{\omega}_{\alpha,I}$. We define now:

$$F_{\alpha,I} := \int_F \tilde{\omega}_{\alpha,I}.$$

For each U_α we define the integration along the fiber of ω to be the form:

$$\tau_\alpha = \sum_{\#I=p-r} F_{\alpha,I} dx_\alpha^I.$$

Again, thanks to equation (1.3) we have that the τ_α's glue together to a well defined $(p-r)$-form on M, that we shall denote by $\pi_*(\omega)$.

Proposition 1.5.2 (Projection formula)**.** *Let M be a smooth manifold of dimension n and let $\pi : B \to M$ be an oriented fiber bundle with fiber a compact oriented manifold F of dimension r (possibly with boundary).*

1. *For $\omega \in A^p(B)$ and $\theta \in A^q(M)$,*

$$\pi_*(\omega \wedge \pi^*\theta) = \pi_*\omega \wedge \theta.$$

2. *if M is compact and oriented, for ω in $A^p(B)$ and θ in $A^{n+r-p}(M)$*

$$\int_B \omega \wedge \pi^*\theta = \int_M \pi_*\omega \wedge \theta.$$

Proof. First of all, given a form θ in $A^q(M)$, we know that on each U_α its pull-back to B is a form constant along the fibers and having only dx_α components. On $\pi^{-1}U_\alpha$ we see that $\pi^*\theta|_{\pi^{-1}(U_\alpha)} = \sum_J g_J(x_\alpha) dx_\alpha^J$, where J is a q-uple of integers. Therefore, if ω is a form in $A^p(B)$, having as components involving dt^λ on each $W_{\alpha,\lambda}$

$$\sum_I f_{\alpha,I}(x_\alpha, t_\lambda) dt^\lambda \wedge dx_\alpha^I$$

and

$$\omega \wedge \pi^*\theta|_{W_{\alpha,\lambda}} = \sum_{I,J} f_{\alpha,I}(x_\alpha, t_\lambda) dt^\lambda \wedge dx_\alpha^I \wedge g_J(x_\alpha) dx_\alpha^J + \text{terms without } dt^\lambda.$$

Integrating $f_{\alpha,I}(x_\alpha, t_\lambda) dt^\lambda$ we get the assertion. The second part of the proposition is proved by direct computation; if we denote by $\{\rho_\alpha\}$ a partition of unity adapted to $\{U_\alpha\}$ we have:

$$\int_B \omega \wedge \pi^*\theta = \sum_\alpha \int_F \int_{U_\alpha} \pi^*\rho_\alpha \cdot \omega \wedge \pi^*\theta|_{\pi^{-1}(U_\alpha)}.$$

Since $\pi^*\theta$ has no dt_λ components, we have that:

$$\sum_\alpha \int_F \int_{U_\alpha} \pi^*\rho_\alpha \cdot \omega \wedge \pi^*\theta|_{\pi^{-1}(U_\alpha)} = \sum_\alpha \int_{U_\alpha} \rho_\alpha \pi_*\omega \wedge \theta|_{U_\alpha} = \int_M \pi_*\omega \wedge \theta.$$

□

The second important proposition is the following.

Proposition 1.5.3. *Let M be a smooth manifold of dimension n and let $\pi : B \to M$ be an oriented fiber bundle with fiber a compact oriented manifold F of dimension r (possibly with boundary). Then:*

$$\pi_* \circ d + (-1)^{r+1} d \circ \pi_* = (\partial\pi)_* \circ i^*,$$

where i is the inclusion $\partial B \hookrightarrow B$.

Proof. We take a partition of unity adapted to $\{W_{\alpha,\lambda}\}$. Since both d and π_* are linear we can use a partition of unity argument and prove the assertion for each coordinate patch $\{W_{\alpha,\lambda}\}$. We first prove the assertion for a coordinate patch $W_{\alpha,\lambda}$ such that $W_{\alpha,\lambda} \cap \partial B = \emptyset$. Then the assertion becomes:

$$\pi_* \circ d = (-1)^r d \circ \pi_*.$$

We prove it first for forms which can be expressed as $\sum_I f_{\lambda,I} dt^\lambda \wedge dx^I_\alpha$. Please note that $i^* dt^\lambda = 0$, so, for such a form this relation would hold also if $W_{\alpha,\lambda} \cap \partial B \neq \emptyset$. Now:

$$\begin{aligned}
\pi_* \circ d(\sum_I f_{\lambda,I} dt^\lambda \wedge dx^I_\alpha) &= \pi_*\left(\sum_I \sum_{i=1}^n \frac{\partial f_{\lambda,I}}{\partial x^i_\alpha} dx^i_\alpha \wedge dt^\lambda \wedge dx^I_\alpha\right) \\
&= \sum_I \sum_{i=1}^n (-1)^r \pi_*\left(\frac{\partial f_{\lambda,I}}{\partial x^i_\alpha} dt^\lambda \wedge dx^i_\alpha \wedge dx^I_\alpha\right) \\
&= \sum_I \sum_{i=1}^n (-1)^r \pi_*\left(\frac{\partial f_{\lambda,I}}{\partial x^i_\alpha} dt^\lambda\right) \wedge dx^i_\alpha \wedge dx^I_\alpha \\
&= (-1)^r d \circ \pi_*\left(\sum_I f_{\lambda,I} dt^\lambda \wedge dx^I_\alpha\right).
\end{aligned}$$

Now, we prove it for forms of type

$$\sum_I \sum_{\nu=1}^r f_{\lambda,I} dt^1_\lambda \wedge \ldots \wedge \hat{dt}^\nu_\lambda \wedge \ldots \wedge dt^r_\lambda \wedge dx^I_\alpha.$$

Please note that π_* of such a form is 0. Suppose now that $W_{\alpha,\lambda} \cap \partial B = \emptyset$, then, for this type of form, we have that the left hand side is 0 because of Stokes theorem and the right hand side is 0 since in this coordinate patch i is the immersion of the empty set. Without loss of generality we can suppose that $W_{\alpha,\lambda} \cap \partial B = \{t^r_\alpha = 0\}$. We have that since we are integrating the differential of this form along the fiber, on each fiber, by Stoke's theorem this is equivalent to integrate the form on the boundary $W_{\alpha,\lambda} \cap \partial B = \{t^r_\alpha = 0\}$. So, without loss of generality, we can suppose the form is of type

$$\sum_I f_{\lambda,I} dt^1_\lambda \wedge \ldots \wedge dt^{r-1}_\lambda \wedge dx^I_\alpha.$$

For forms of this type the relation is easily seen to be satisfied. □

Now, suppose we have an oriented real vector bundle $\pi : E \to M$ of rank r over a smooth manifold M of dimension n. We identify M with the zero section of E. We set $W_0 = E \setminus M$ and $W_1 = E$ and we consider the Čech-de Rham cohomology with respect to the covering $\mathcal{W} = \{W_0, W_1\}$. Let T_1 be a closed disc bundle in W_1 and $T_0 = E \setminus \operatorname{Int} T_1$. Then $\{T_0, T_1\}$ is a system of honeycomb cells adapted to $\mathcal{W}$. If we denote by π_1 the restriction of π to T_1 and by π_{01} the restriction to T_{01} we define the "integration along the fiber"

$$\pi_* : A^p(\mathcal{W}, W_0) \to A^{p-r}(M)$$

by:

$$(0, \sigma_1, \sigma_{01}) \to (\pi_1)_* \sigma_1 + (\pi_{01})_* \sigma_{01}.$$

From the propositions we proved above we have the following.

Lemma 1.5.4. *Let M be a smooth manifold of dimension n and let $\pi : E \to M$ be a smooth, oriented, real vector bundle of rank r. Then:*

1. *for σ in $A^p(\mathcal{W}, W_0)$ and θ in $A^q(M)$,*
$$\pi_*(\sigma \smile \pi^*\theta) = \pi_*\sigma \wedge \theta,$$
where we consider $\pi^*\theta$ as an element in $A^q(W_1)$;
2. *if M is compact and oriented, then for σ in $A^p(\mathcal{W}, W_0)$ and θ in $A^{n+r-p}(M)$,*
$$\int_E \sigma \smile \pi^*\theta = \int_M \pi_*\sigma \wedge \theta;$$
3. $$\pi \circ D + (-1)^{r+1} d \circ \pi_* = 0,$$
where by $d \circ \pi_*$ we mean $d \circ (\pi_1)_* + d \circ (\pi_{01})_*$.

Using this last proposition one can prove that the integration along the fiber induces a homomorphism in cohomology.

Theorem 1.5.5 (Thom isomorphism)**.** *The homomorphism induced by integration along the fiber:*
$$\pi_* : H^p(E, E \setminus M; \mathbb{C}) \to H^{p-r}(M; \mathbb{C})$$
is an isomorphism.

For a proof of this theorem we refer to [11] or [31]. The proof of this theorem permits us to find an inverse T_E for π_* and a class ϕ_E (depending on the vector bundle E), called the **Thom class**, which corresponds to the class [1] in $H^0(M; \mathbb{C})$. In general:
$$T_E(a) = \phi_E \smile \pi^* a.$$

Suppose now M is a smooth manifold of dimension n and S is a smooth submanifold of dimension k. There exists an interesting relation between Poincaré and Alexander duality and Thom isomorphism. Since S is a submanifold it admits a tubular neighborhood T_ϵ which is diffeomorphic to the normal bundle $N_{S/M}$. We consider now the Thom isomorphism $\pi_* : H^{p+r}(T_\epsilon, T_\epsilon \setminus S; \mathbb{C}) \to H^p(S; \mathbb{C})$. Thanks to the excision principle, this isomorphism induces an isomorphism $\pi_* : H^{p+r}(M, M \setminus S; \mathbb{C}) \to H^p(S; \mathbb{C})$. Now, r is exactly the codimension of S in M, i.e., $r = n - k$. Thanks to Lemma 1.5.4 we have the following commutative diagram:

$$\begin{tikzcd}
H^{p+r}(M, M \setminus S; \mathbb{C}) \arrow[r, "\simeq"] \arrow[d, "A_{M_S}"'] & H^{p+r}(T_\epsilon, T_\epsilon \setminus S) \arrow[r, "\pi_*"] & H^p(S; \mathbb{C}) \arrow[d, "P_S"] \\
H_{k-p}(S; \mathbb{C}) \arrow[rr, "\simeq"] & & H_{k-p}(S; \mathbb{C}).
\end{tikzcd}$$

This diagram tells us that the Thom class corresponds to the Alexander dual of the class [1] in $H_0(S; \mathbb{C})$; moreover, thanks to the long exact sequence in cohomology for the couple $(M, M \setminus S)$ we can extend a class in $H^r(M, M \setminus S; \mathbb{C})$ to a class of $H^r(M; \mathbb{C})$ with support in a tubular neighborhood of S; hence, the Thom class is a good tool to compute the Poincaré dual of a submanifold.

1.6 Chern classes in the Chern-Weil framework

Let E be a complex vector bundle on a differentiable manifold M, we denote by $A^p(M,E)$ the set of p-forms with values in E and with $A^p(M)$ the p-forms on M, where A^0 is understood to be the sheaf of smooth functions $A^0(M) = C^\infty(M)$. We refer to [18] and [31] for a treatment of this topic.

Definition 1.6.1. Let $\pi : E \to M$ be a complex vector bundle over a differentiable manifold M. A **connection** on E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M,E) \to A^1(M,E)$$

which satisfies the Leibniz rule, i.e.

$$\nabla(f \cdot s) = df \otimes s + f\nabla(s) \quad \text{for } f \in A^0(M),\ s \in A^0(E).$$

Lemma 1.6.2. *A connection ∇ is a local operator: if a section s of E is identically 0 on an open set U, so is $\nabla(s)$.*

Proof. Let p be a point in U and ϕ a function which is 0 on a neighborhood V of p contained in U and such that $\phi \equiv 1$ outside U. Then $s = \phi \cdot s$ everywhere on M. Now:

$$\nabla(s)(p) = \nabla(\phi \cdot s)(p) = d\phi \otimes s(p) + \phi(p) \cdot \nabla(s)(p).$$

Since ϕ is constant on V then $d\phi(p) = 0$ on V. So $\nabla(s)(p) = 0$. Since this is true for every point p in U we have that $\nabla(s) = 0$ on U. □

So we can restrict a connection to an open subset U of M. In particular, if we have a trivializations $\{U_i\}$ of E on M it makes sense to look at the restriction of the connection to U_i.

Lemma 1.6.3. *Let $\nabla_1, \dots, \nabla_k$ connections for E and $\psi_1, \dots, \psi_k$ functions on M with $\sum_{i=1}^k \psi_i \equiv 1$. Then $\sum_{i=1}^k \psi_i \nabla_i$ is a connection for E.*

Proof. We have to check the definition. $\sum_{i=1}^k \psi_i \nabla_i$ is clearly a linear map from $A^0(E,M)$ to $A^1(E,M)$. Furthermore

$$\sum_{i=1}^k \psi_i \nabla_i (f \cdot s) = \sum_{i=1}^k \psi_i \cdot df \otimes s + f \cdot \sum_{i=1}^k \psi_i \nabla_i(s)$$

$$= df \otimes s + f \cdot \sum_{i=1}^k \psi_i \nabla_i(s).$$

□

Suppose now we have a vector bundle E of rank k with trivializations $\{(U_i, \phi_i)\}$. Let s be a section of E; on U_i we have that $\phi_i(s|_{U_i}) = (s^1, \dots, s^k)$. On U_i we can define a connection ∇_i by sending $(s^1, \dots, s^k)$ in $(ds^1, \dots, ds^k)$ (this is called the **flat connection** on U_i). If $\{e_{i,1}, \dots, e_{i,n}\}$ is the frame for $E|_{U_i}$ given by ϕ_i the action of ∇ on this frame is nothing else but $\nabla_i(e_{i,j}) = 0$ for each $j = 1, \dots, n$. Let $\{\psi_i\}$ be a partition of unit subordinated to $\{U_i\}$. Since the sum is locally finite, by the lemma $\sum_i \phi_i \nabla_i$ is a connection on E. This construction shows that every vector bundle admits a connection.

Remark 1.6.4. Please note that we did not define a global flat connection. In fact, let U_α and U_β be two sets of the trivialization with non empty intersection. Let $\{e_{\alpha,1},\dots,e_{\alpha,n}\}$ and $\{e_{\beta,1},\dots,e_{\beta,2}\}$ be frames for $E|_{U_\alpha}$ and $E|_{U_\beta}$ respectively. Then $e_{\beta,i}=(g_{\beta\alpha})_i^j e_{\alpha,j}$, where the $g_{\alpha\beta}$ are the transition functions for the vector bundle. Let ∇ be the flat connection on U_α with respect to the frame $\{e_{\alpha,1},\dots,e_{\alpha,n}\}$. On $U_\alpha\cap U_\beta$ we want to look at the behaviour of the connection when we change frame:

$$\begin{aligned}\nabla e_{\beta,i} &= \nabla\Bigg(\sum_{j=1}^{n}(g_{\beta\alpha})_i^j e_{j,\alpha}\Bigg)=\sum_{j=1}^{n} d(g_{\beta\alpha})_i^j\otimes e_{j,\alpha}+\sum_{j=1}^{n}(g_{\beta\alpha})_i^j\nabla(e_{j,\alpha})\\ &=\sum_{j,i'=1}^{n} d(g_{\beta\alpha})_i^j(g_{\beta\alpha}^{-1})_j^{i'}\otimes e_{i',\beta}. \qquad (1.4)\end{aligned}$$

While the connection was flat with respect to the frame of $E|_{U_\alpha}$, it may not be flat anymore on $U_\alpha\cap U_\beta$ with respect to the frame of $E|_{U_\beta}$.

We are now interested in giving a local representation of a connection.

Lemma 1.6.5. *If ∇ and ∇' are two connections on a vector bundle E then $\nabla-\nabla'$ is $A^0(M)$-linear and thus it can be considered an element in $A^1(M,End(E))$. If ∇ is a connection on E and A is an element in $A^1(M,End(E))$, then $\nabla+\theta$ is again a connection on E*

Proof. By direct computation:

$$\nabla(f\cdot s)-\nabla'(f\cdot s)=df\otimes s+f\nabla(s)-df\otimes s-f\nabla'(s)=f\cdot(\nabla(s)-\nabla'(s)).$$

Let now U_i be a trivializing open subset of E. Then an element A in $A^1(M,\mathrm{End}(E))$ can be represented in this trivialization as a matrix of 1-forms (θ_i^j). Let s be a section of E, in the trivialization we have that $\phi_i\circ s|_{U_i}=(s^1,\dots,s^n)$. The action of A on s can be then described in the trivialization as:

$$(A\cdot s)^j=\sum_{i=1}^{k}\theta_i^j s^i,$$

which is clearly an element of $A^1(M,E)$. Let now A be in $A^1(M,\mathrm{End}(E))$; then

$$\nabla(f\cdot s)+A(f\cdot s)=df\otimes s+f\cdot\nabla(s)+f\cdot A(s)=df\otimes s+f(\nabla+A)(s).$$

□

Remark 1.6.6. We have seen that on each set of a trivialization for E we can define a flat connection. On each trivialization U_i we can represent ∇ in the form $d+\theta$ where θ is in $A^1(M,\mathrm{End}(E)|_{U_i})$.

Let now ∇ be a connection for E, we define its extension ∇ from $A^p(M,E)$ to $A^{p+1}(M,E)$. Let μ be a p-form with values in E, In a trivialization for $A^p(M,E)$ it can be written as sum of forms of the type $\omega\otimes s$ where ω is a p-form on M and s is a section of E. We define then

$$\nabla : A^p(M,E)\to A^{p+1}(M,E)$$

as

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla(s).$$

Moreover, a generalized Leibniz rule holds for this extension, i.e., for any section α of $A^p(M, E)$ and for any k-form β one has:

$$\nabla(\beta \wedge \alpha) = d(\beta) \wedge \alpha + (-1)^k \beta \wedge \nabla(\alpha),$$

since, if $\alpha = \omega \otimes s$ then

$$\begin{aligned}\nabla(\beta \wedge \alpha) &= \nabla((\beta \wedge \omega) \otimes s) = d(\beta \wedge \omega) \otimes s + (-1)^{p+k}(\beta \wedge \omega) \otimes \nabla(s)\\ &= d(\beta) \wedge \omega \otimes s + (-1)^p((\beta \wedge d(\omega)) \otimes s + (-1)^k(\beta \wedge \omega) \otimes \nabla(s))\\ &= d(\beta) \wedge \alpha + (-1)^p \beta \wedge \nabla(\alpha).\end{aligned}$$

Definition 1.6.7. The composition $\Omega := \nabla \circ \nabla$ from $A^0(M, E)$ to $A^2(M, E)$ is called the **curvature** of ∇.

Given a section s of E and a function $f \in A^0(M)$:

$$\begin{aligned}\nabla^2(f \cdot s) =& \nabla(df \otimes s + f \cdot \nabla(s)) =\\ =& d^2(f) \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\nabla^2(s) = f \cdot \nabla^2(s).\end{aligned}$$

So the curvature is a $A^0(M)$ linear operator on E and we can consider it as an element of $A^2(M, \mathrm{End}(E))$.

Since a connection on E can be represented as $d + \theta$ with θ in $A^1(M, E)$ we would like to give a description of Θ in terms of θ. We have then in a trivialization, supposing E has rank r:

$$\begin{aligned}\nabla^2\Big(e_i\Big) =& \nabla\Big(\sum_{j=1}^r (\theta_i^j) e_j\Big)\\ =& \sum_{i=1}^r \sum_{j=1}^r d(\theta_i^j) \otimes e_j - \sum_{j=1}^r \sum_{k=1}^r \theta_i^j \wedge \theta_j^k \otimes e_k,\end{aligned}$$

so $\Theta = d\theta + \theta \wedge \theta$, where by $\theta \wedge \theta$ we mean $(\theta \wedge \theta)_i^j := \sum_{k=1}^n \theta_k^j \wedge \theta_i^k$.

Suppose now E_1 and E_2 are vector bundles with connections ∇_1 and ∇_2. We can define a connection on $E_1 \otimes E_2$ by $\nabla(s_1 \otimes s_2) := \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$. On $E_1 \oplus E_2$ we can define a connection ∇ by $\nabla(s_1 \oplus s_2) := \nabla_1(s_1) \oplus \nabla_2(s_2)$. If E is a vector bundle with connection ∇ we can define a connection ∇^* on its dual by

$$\nabla^*(\omega)(s) = d(\omega(s)) - \omega(\nabla(s)),$$

where by $\omega(\nabla(s))$ we mean the extension by 1-form linearity of ω, i.e.,

$$\omega(\sum_i \gamma^i \otimes e_i) = \sum_i \gamma^i \omega(e_i),$$

where for each i, the γ^i's are 1-forms.

Suppose now we have a trivialization of E, let $(e_1, \ldots, e_r)$ be trivializing sections. Then we can express the connection ∇ on E as $d + \theta$ where θ is a matrix of one forms. Let $(\omega^1, \dots, \omega^r)$ be the dual frame for E^*. We want to find the connection matrix for ∇^* given the connection matrix θ of ∇:

$$\nabla^*(\omega^i)(e_j) = d(\omega^i(e_j)) - \omega^i(\nabla(e_j)) = d(\delta^i_j) - \omega^i\left(\sum_{l=1}^r \theta^l_j \cdot e_l\right) = -\theta^i_j.$$

Therefore

$$\nabla^*(\omega^i) = -\sum_{j=1}^r \theta^i_j \omega^j.$$

So the connection matrix of ∇^* is $-\theta^t$ and we can locally express ∇^* as $d - \theta^t$. We can now express the connection on the endomorphism bundle $\mathrm{End}(E, E) = E^* \otimes E$; on a trivializing neighborhood

$$\nabla(\omega^j \otimes e_i) = \nabla^*(\omega^j) \otimes e_i + \omega^j \otimes \nabla(e_i) = \sum_{l=1}^r -\theta^j_l \omega^l \otimes e_i + \sum_{k=1}^r \omega^j \otimes \theta^k_i e_k.$$

Therefore for an endomorphism $(B^i_j \omega^j \otimes e_i)$ we have

$$\nabla(\sum_{i,j=1}^r B^i_j \omega^j \otimes e_i) = \sum_{i,j=1}^r dB^i_j \otimes \omega^j \otimes e_i - \sum_{i,j,k=1}^r B^i_j \theta^j_k \omega^k \otimes e_i + \sum_{i,j,l=1}^r B^i_j \theta^l_i \omega^j \otimes e_l.$$

We write this connection in the compact form $d + [\theta, \cdot]$.

Lemma 1.6.8. *Let A be a section of the endomorphism bundle of E and s a section of E. Then*

$$\nabla(A \cdot s) = \nabla(A) \cdot s + A \cdot \nabla(s).$$

Proof. We compute first

$$\nabla(A \cdot s) = \nabla(\sum_{i,j} s^j A^i_j e_i) = \sum_{i,j} (ds^j A^i_j + s^j dA^i_j + \sum_l s^j A^l_j \theta^i_l) \otimes e_i.$$

Now

$$A \cdot \nabla(s) = A \cdot (\sum_k ds^k \otimes e_k + \sum_{j,k} s^j \theta^k_j e_k) = \sum_{i,j} ds^j A^i_j \otimes e_i + \sum_{i,j,l} s^j \theta^l_j A^i_l \otimes e_i,$$

while

$$\nabla(A) \cdot s = \sum_{i,j} s^j (dA^i_j + \sum_l (A^l_j \theta^i_l - \theta^l_j A^i_l)) \otimes e_i.$$

Comparing the different expressions we get that:

$$\nabla(A \cdot s) = \nabla(A) \cdot s + A \cdot \nabla(s).$$

□

Remark 1.6.9. Using the extensions of the connection for forms and endomorphisms defined above, from last Lemma follows the expression

$$\nabla(\Theta(s)) = \nabla(\Theta)(s) + (-1)^2 \Theta(\nabla(s)).$$

Lemma 1.6.10. *If Θ in $A^2(M, End(E))$ is the curvature of a connection ∇ on a vector bundle E, then*

$$\nabla(\Theta) = 0.$$

Proof. By direct computation, since $\nabla(\Theta(s)) = \nabla(\Theta)(s) + \Theta(\nabla(s))$:

$$\nabla(\Theta)(s) = \nabla(\Theta(s)) - \Theta(\nabla(s)) = \nabla(\nabla^2(s)) - \nabla^2(\nabla(s)) = 0.$$

□

Let V be a complex space of dimension n. A k-multilinear symmetric map $\phi : V \otimes \ldots \otimes V \to \mathbb{C}$ is an element in the k-th symmetric tensor power of V^*. To each of these maps we can associate its polarized form $\tilde{\phi} : V \to \mathbb{C}$ defined by $\tilde{\phi}(v) = \phi(v, \ldots, v)$, for every v in V. Let now ϕ be an element of $\mathrm{Sym}_2(V^*)$. We choose a basis $e_1, \ldots, e_n$ of V and a basis $\omega^1, \ldots, \omega^n$ of V^*. Let $\phi := \sum_{i,j=1}^n \omega^i \otimes \omega^j$ and let $v = (v_1, \ldots, v_n)$ be a vector in V, then $\tilde{\phi}(v) = \sum_{i,j=1}^n v_i \otimes v_j$ is a homogeneous polynomial of degree 2 in the coordinates of v.

In general if we take ϕ in $\mathrm{Sym}_k(V^*)$ and evaluate its polarized form on the vector $\lambda \cdot v$:

$$\tilde{\phi}(\lambda \cdot v) = \phi(\lambda \cdot v, \ldots, \lambda \cdot v) = \lambda^k \phi(v, \ldots, v) = \lambda^k \tilde{\phi}(v).$$

Since a k-multilinear form is a polynomial in the coordinates of the vectors on which it is evaluated we get that the polarized form of a ϕ multilinear symmetric map is a homogeneous polynomial. There is a one to one correspondence between multilinear symmetric maps and homogeneous polynomials; indeed we can construct a symmetric k-multilinear map from a homogeneous polynomial of degree k by means of the polarization formula.

We consider now the case of $V = \mathfrak{gl}(r, \mathbb{C}) \simeq \mathrm{Mat}_r(\mathbb{C})$. Let ϕ be a k-multilinear map on $\mathfrak{gl}(r, \mathbb{C})$. Such a map is called **invariant** if for all G in $\mathrm{GL}(r, \mathbb{C})$ and all $A_1, \ldots, A_k$ in $\mathfrak{gl}(r, \mathbb{C})$ one has:

$$\phi(G \cdot A_1 \cdot G^{-1}, \ldots, G \cdot A_k \cdot G^{-1}) = \phi(A_1, \ldots, A_k). \tag{1.5}$$

For every B in $\mathfrak{gl}(r, \mathbb{C})$ and for every t in $\mathbb{R}$ the **exponential of** B given by

$$e^{tB} = I + tB + \frac{1}{2} t^2 \cdot B^2 + \ldots$$

is a well defined matrix in $\mathrm{GL}(r, \mathbb{C})$. We want to differentiate $e^{tB} A e^{-tB}$ with respect to t in 0:

$$\lim_{h \to 0} \frac{e^{hB} A e^{-hB} - A}{h} = \lim_{h \to 0} \frac{(I + hB + o(h^2))A(I - hB + o(h^2)) - A}{h} =$$

$$= \lim_{h \to 0} \frac{A + h \cdot BA - h \cdot AB + o(h^2) - A}{h} = AB - BA = [A, B].$$

Since the exponential of matrices, for fixed t different from 0, is surjective from $\mathfrak{gl}(r, \mathbb{C})$ to $\mathrm{GL}(r, \mathbb{C})$, differentiating both terms of (1.5), we get that the condition for the invariance of ϕ can be rewritten as:

$$\sum_i^k \phi(A_1, \ldots, A_{j-1}, [A_j, B], A_{j+1}, \ldots, A_k)) = 0,$$

for all $B, A_1, \ldots, A_k$ in $\mathfrak{gl}(r, \mathbb{C})$.

Lemma 1.6.11. *Let ϕ be a symmetric invariant k-form on $\mathfrak{gl}(r,\mathbb{C})$. Given a vector bundle E of rank r on a manifold M and fixed an integer m, for any partition $i_1 + i_2 + \ldots + i_k = m$ there exists a naturally induced k-linear map:*

$$\phi : \left(\bigwedge^{i_1} T^*M \otimes End(E)\right) \times \ldots \times \left(\bigwedge^{i_k} T^*M \otimes End(E)\right) \to \bigwedge^{m}_{\mathbb{C}} T^*M$$

defined by $\phi(\alpha_1 \otimes t_1, \ldots, \alpha_k \otimes t_k) = (\alpha_1 \wedge \ldots \wedge \alpha_k)\phi(t_1, \ldots, t_k)$.

Proof. Let U_i be a trivialization for E. Then, on a trivializing neighborhood U_i, the definition of the map makes sense. Since ϕ is invariant, then the map does not depend on the chosen trivialization. $\square$

Then the map induces a k-multilinear map on the level of global sections:

$$\phi : A^{i_1}(M, \mathrm{End}(E)) \times \ldots \times A^{i_k}(M, \mathrm{End}(E)) \to A^m_{\mathbb{C}}(M).$$

If we restrict ϕ to $A^2(M, \mathrm{End}(E)) \times \ldots \times A^2(M, \mathrm{End}(E))$ this induced map is symmetric. So we apply it to the curvature form Θ associated to a connection ∇ on a vector bundle E. We look now to the differential of $\phi(\Theta, \ldots, \Theta)$. By how we extended ϕ to the level of global sections we get that:

$$d\phi(\gamma_1, \ldots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} \phi(\gamma_1, \ldots, d\gamma_j, \ldots, \gamma_k).$$

So if we evaluate it on Θ we get:

$$d\phi(\Theta, \ldots, \Theta) = \sum_{j=1}^{k} \phi(\Theta, \ldots, d\Theta, \ldots, \Theta).$$

Since the connection induced on $A^2(M, \mathrm{End}(E))$ by ∇ can be written as $\nabla = d + [\theta, \cdot\,]$, it acts on the curvature as $\nabla(\Theta) = d\Theta + [\theta, \Theta]$ and we get that:

$$\begin{aligned} d\phi(\Theta, \ldots, \Theta) &= \sum_{j=1}^{k} \phi(\Theta, \ldots, d\Theta, \ldots, \Theta) \\ &= \sum_{j=1}^{k} \phi(\Theta, \ldots, \nabla(\Theta) - [\theta, \Theta], \ldots, \Theta); \end{aligned}$$

but $\nabla(\Theta) = 0$ so

$$d\phi(\Theta, \ldots, \Theta) = \sum_{j=1}^{k} \phi(\Theta, \ldots, [\theta, \Theta], \ldots, \Theta);$$

using the invariance of ϕ we have that $d\tilde{\phi}(\Theta) = 0$. Therefore $\tilde{\phi}(\Theta)$ defines a class in de Rham cohomology. The open question now is whether this class depends on the chosen connection; this issue is solved by the existence of an important object, the Bott difference form.

In the following, if ϕ is a symmetric polynomial of degree n and ∇ is a connection with curvature form Θ we denote by $\phi(\nabla) := \phi(\Theta)$.

Lemma 1.6.12 (Existence of Bott's form)**.** *Given a symmetric polynomial ϕ of degree n and $p+1$ connections $\nabla_0, \dots, \nabla_p$ there exists a form $\phi(\nabla_0, \dots, \nabla_p)$ of degree $2n-p$, alternating in the $p+1$ entries and satisfying*

$$\sum_{i=0}^{p}(-1)^i \phi(\nabla_0, \dots, \hat{\nabla}_i, \dots, \nabla_p) + (-1)^p d\phi(\nabla_0, \dots, \nabla_p) = 0. \tag{1.6}$$

Proof. Consider the vector bundle $E \times \mathbb{R}^{p+1} \to M \times \mathbb{R}^{p+1}$ and let

$$\tilde{\nabla} = \left(1 - \sum_{i=1}^{p} t_i\right)\nabla_0 + \sum_{i=1}^{p} t_i \nabla_i.$$

If we denote by Δ^p the standard p-simplex in $\mathbb{R}^{p+1}$, i.e.,

$$\Delta^p = \{(t_0, \dots, t_p) \in \mathbb{R}^{p+1} \mid \sum_{i=0}^{p} t_i = 1\}$$

and we denote by

$$\pi : M \times \Delta^p \to M$$

the projection, we can integrate along the fiber $\pi_* : A^*(M \times \Delta^p) \to A^{*-p}(M)$. We set $\phi(\nabla_0, \dots, \nabla_p) = \pi_*(\phi(\tilde{\nabla}))$. Now, integration along the fiber has an important property (proved above in 1.5.4); we have that for a form ω in $A^n(B)$:

$$\pi_* \circ d(\omega) + (-1)^{p+1} d \circ \pi_*(\omega) = (\partial\pi)_* \circ i^*(\omega),$$

where i is the inclusion $\partial B \to B$ and $(\partial\pi)_*$ is the integration along the boundary of the fiber.

We know that $\phi(\tilde{\nabla})$ is a closed form and therefore

$$\pi_*(d\phi(\nabla)) = 0;$$

as ν varies in $0, \dots, p$ we denote by i_ν the inclusion of the face $\Delta^{p-1}_\nu = \{p \in \Delta^p \subset \mathbb{R}^p \mid t_\nu = 0\}$ in Δ^p; if we denote by

$$\tilde{\nabla}_\nu = \tilde{\nabla}|_{t_\nu = 0};$$

since on Δ^{p-1}_ν we have that $\sum_{i=1, i\neq\nu}^{p} t_i = 1$

$$\tilde{\nabla}_\nu = \left(1 - \sum_{i=1, i\neq\nu}^{p} t_i\right)\nabla_0 + \sum_{i=1, i\neq\nu}^{p} t_i \nabla_i,$$

then

$$i^*_\nu \phi(\tilde{\nabla}) - \phi(\tilde{\nabla}_\nu) = 0,$$

because of the naturality of the pull-back with respect to all the operations with forms. If we denote by $\partial\pi_{\nu,*}$ the integration along the face Δ^{p-1}_ν we have that

$$\phi(\nabla_0, \dots, \widehat{\nabla_\nu}, \dots, \nabla_p) = (\partial\pi)_{\nu,*}(\phi(\tilde{\nabla}_\nu)).$$

Therefore:

$$(-1)^{p+1} d \circ \pi_*(\phi(\tilde{\nabla})) = (\partial\pi)_* \circ i^*(\phi(\tilde{\nabla})) = (\partial\pi)_*(\sum_{\nu=0}^{p}(-1)^\nu i^*_\nu(\phi(\tilde{\nabla})))$$

$$= \sum_{\nu=0}^{p}(-1)^\nu \phi(\nabla_0, \dots, \widehat{\nabla_\nu}, \dots, \nabla_p).$$

□

Therefore, given two connections ∇_0 and ∇_1 for a vector bundle E, the difference $\phi(\nabla_0) - \phi(\nabla_1) = d\phi(\nabla_0, \nabla_1)$ is exact. Then the class $[\phi(\nabla)]$ in de Rham cohomology does not depend on the connection chosen and the following definition makes sense.

Definition 1.6.13. Let E be a complex vector bundle on a complex manifold M; let ϕ be a homogeneous polynomial of degree d. For any connection ∇ on E let $[\phi(\nabla)]$ be the class in de Rham cohomology $H^{2d}(M, \mathbb{R})$ of the closed $2d$-form $\phi(\nabla)$. We denote this class by $\phi(E)$ and call it the **characteristic class of E for the polynomial ϕ**.

We have now a nice recipe to calculate some invariants of M. Clearly, some homogenous polynomials will be more interesting than others. Indeed, let A be a diagonalizable matrix in $\mathrm{GL}(r, \mathbb{C})$. We know that it is coniugated to a diagonal matrix D, by an invertible matrix B. By Binet's formula

$$\det(A + I) = \det(B^{-1}) \det(D + I) \det(B) = \det(D + I) = \prod_{i=1}^{r}(1 + \lambda_i),$$

where λ_i are the eigenvalues of A. But

$$\prod_{i=1}^{r}(1 + \lambda_i) = 1 + \sigma_1(\lambda_1, \ldots, \lambda_r) + \ldots + \sigma_r(\lambda_1, \ldots, \lambda_r), \tag{1.7}$$

where σ_i are the elementary symmetric polynomials in $\lambda_1, \ldots, \lambda_n$. Since the matrix is diagonalizable the eigenvalues can be expressed as polynomial functions of the coefficients of the matrix and they are clearly invariant by coniugation. Now, the diagonalizable matrices are dense $\mathrm{GL}(n, \mathbb{C})$, so by continuity the equality (1.7) holds in general.

Definition 1.6.14. We call $c(E) = \det(I + \sqrt{-1}/2\pi\Theta)$ the **total Chern class** and each of the $c_i(E) := \sigma_i(E)$ is called the i-th **Chern class** of E.

The Chern classes are functorial [27], indeed, given two vector bundles E_1 and E_2 then $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$. Moreover, suppose we have a trivial bundle E of rank r. If it is trivial we have a globally defined flat connection, with $\Theta = 0$. Since Chern classes do not depend on the connection we have that $c(E) = 0$. Suppose now that $E \equiv E' \oplus T_s$, where T_s is a trivial bundle of rank s. Then $c_j(E) = 0$ for $j = r - s + 1, \ldots, r$.

1.7 Chern classes in Čech-de Rham cohomology

Recall we can define a connection for a vector bundle E on a manifold M defining a connection for E on each trivializing neighborhood and gluing them together to a global connection using a partition of unity. In section 1.4 we saw that Čech-de Rham cohomology behaves naturally with respect to this gluing operation and this makes us think that Čech-de Rham cohomology could be an interesting framework for the theory of Chern classes (the main reference for this section is [31]).

Suppose $\{U_\alpha\}$ is an open cover of a manifold M and let $\pi : E \to M$ be a complex vector bundle of rank r and let ψ be a homogeneous symmetric polynomial of degree d. For each α we choose a connection ∇_α for E on U_α and we define an element $\psi(\nabla_*)$ as a Čech cochain for the sheaf A^{2d} by:

$$\psi(\nabla_*)_{\alpha_0\ldots\alpha_p} = \psi(\nabla_{\alpha_0}, \ldots, \nabla_{\alpha_p}).$$

Now, from property (1.6) we have that $D\psi(\nabla_*) = 0$, so we have defined a cocycle in Čech-de Rham cohomology. We check now this cocycle is well defined; if we take another collection of connections ∇'_* and define a new cochain ϕ by

$$\phi_{\alpha_0\ldots\alpha_p} = \sum_{\nu=0}^{p} (-1)^\nu \psi(\nabla_{\alpha_0}, \ldots, \nabla_{\alpha_\nu}, \nabla'_{\alpha_\nu}, \ldots, \nabla'_{\alpha_p}),$$

we have that

$$\psi(\nabla'_*) - \psi(\nabla_*) = D\phi.$$

Hence, $[\psi(\nabla_*)]$ is a well defined class in Čech-de Rham cohomology. Moreover, if we take the image of this class through the isomorphism between Čech-de Rham and de Rham cohomology we see that the class $[\psi(\nabla_*)]$ corresponds to the class $\phi(E)$ [31].

1.8 Foliations and coherent sheaves: definitions and basic notions

In this section we will define some of the basic objects of our treatment and define the cathegory in which our objects naturally live, the cathegory of coherent sheaves. We refer to [20] for a treatment of coherent sheaves. It is known that there is a 1-1 relationship between locally free sheaves and vector bundles on complex analytic manifolds, i.e., locally free sheaves are the sheaves of section of vector bundles. Coherent sheaves are an object more general than locally free sheaves while retaining most of their interesting properties; intuitively one could think of the coherent sheaves on a complex manifold M as sheaves which are locally free outside an analytic subset Σ in M.

Definition 1.8.1. Let M be a complex manifold and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_M$-modules on M. We say $\mathcal{E}$ is **finitely generated** if, for every point x in M, there exist an open neighborhood U_x, a free module $\mathcal{O}_M^n$ of finite rank n and a surjective mapping $\pi : \mathcal{O}_M^n\mid_U \to \mathcal{E}\mid_U$. A finitely generated sheaf $\mathcal{E}$ is **coherent** if for any U open set in M and any morphism $\phi : \mathcal{O}_M^k\mid_U \to \mathcal{E}\mid_U$ of $\mathcal{O}_M$-modules the kernel of ϕ is finitely generated. If $\mathcal{E}$ is a coherent sheaf, we define the **support** of $\mathcal{E}$ as $\mathrm{Supp}(\mathcal{E}) := \{x \in M \mid \mathcal{E}_x \neq 0\}$.

On a general topological space, by a basic result of Serre, if two of the sheaves of $\mathcal{O}_M$-modules in a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$$

are coherent then so is the third; moreover, given two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ we have that $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is coherent.

We make more precise the idea of a coherent sheaf as a locally free sheaf outside an analytic subset. Indeed, the definition we gave of coherent sheaf is equivalent to the fact that, given a coherent sheaf $\mathcal{E}$ for every x in M there exists an open set U_x and an exact sequence:

$$\mathcal{O}_U^q \xrightarrow{f} \mathcal{O}_U^p \xrightarrow{\pi} \mathcal{E}\,|_U \longrightarrow 0,$$

with p and q positive integers.

We claim the support of $\mathcal{E}$ is a holomorphic subvariety of M. Indeed the homomorphism f is described by a $q \times p$ matrix of holomorphic functions $(f)_{ij}$ on U_x. If $p > q$ then $f(\mathcal{O}_{U,y}^q)$ is a proper submodule of $\mathcal{O}_{U,y}^p$ and therefore $\mathcal{E} \neq 0$ for every y in U_x. Without loss of generality we can suppose that $p \leq q$. Now, we have that $\mathcal{E}_y = 0$ if and only if $f(\mathcal{O}_{U,y}^q) \supseteq \mathcal{O}_{U,y}^p$. Now, with some work, it is proved that this happens if and only if the matrix $f_{ij}(y)$ has rank p; therefore the support of $\mathcal{E}$ are the points y in U_x where the rank of $f_{ij}(y) < p$; this is the zero set of the determinants of the $p \times p$ minors of $f_{ij}(y)$ and therefore an analytic subset of U_x.

Definition 1.8.2. For a coherent sheaf $\mathcal{E}$ we define

$$\mathrm{Sing}(\mathcal{E}) = \{x \in M \mid \mathcal{E}_x \text{ is not } \mathcal{O}_{M,x}\text{-free}\}$$

and call it the **singular set** of $\mathcal{E}$.

Remark 1.8.3. By definition $\mathcal{E}$ is locally free on $M \setminus \mathrm{Sing}(\mathcal{E})$. Its rank is called the **rank** of $\mathcal{E}$.

Now we can define precisely what a singular holomorphic foliation is.

Definition 1.8.4. (The tangent sheaf) of a **(singular) holomorphic foliation** is a coherent subsheaf $\mathcal{F}$ of $\mathcal{T}_M$, closed with respect to the bracket of vector fields, i.e., for every x in M we have $[\mathcal{F}_x, \mathcal{F}_x] \subseteq \mathcal{F}_x$.

Definition 1.8.5. Let $\mathcal{F}$ be a (singular) holomorphic foliation. We set $\mathcal{N}_\mathcal{F} = \mathcal{T}_M \,/\, \mathcal{F}$ and we denote by $S(\mathcal{F}) := \mathrm{Sing}(\mathcal{N}_\mathcal{F})$ and call it the **singular set of the foliation**.

Remark 1.8.6. From the discussion above it is easily seen that $S(\mathcal{F})$ is an analytic variety containing $\mathrm{Sing}(\mathcal{F})$ as a subset, since $\mathcal{F}$ being non $\mathcal{O}_{M,x}$-free forces $\mathcal{N}_\mathcal{F}$ to be non $\mathcal{O}_{M,x}$-free, while, if $\mathcal{N}_\mathcal{F}$ is free so is $\mathcal{F}$. An example of the situation in which $\mathcal{F}$ is free while $\mathcal{N}_\mathcal{F}$ is not free is when $\mathcal{F}$ is generated on M by a single holomorphic vector field; then $\mathcal{F}$ is $\mathcal{O}_M$-free while $\mathcal{N}_\mathcal{F}$ is not $\mathcal{O}_M$-free on the zero set of v.

Remark 1.8.7. We can describe $S(\mathcal{F})$ more concretely thanks to a couple of remarks; if $\mathcal{F}$ is a coherent subsheaf of $\mathcal{T}_M$ it is locally finitely generated, i.e., for every point x in M there exists a neighborhood U_x of x and sections $v_1, \dots, v_r$ generating $\mathcal{F}$ over U_x (since $\mathcal{F}$ is a subsheaf of $\mathcal{T}_M$ they are vector fields). Without loss of generality we can suppose U is a coordinate neighborhood with coordinates $(z^1, \dots, z^n)$; if we write each v_i as $v_i = \sum_j f_{ij}(z)\partial/\partial z_j$ and since $\mathcal{T}_M\,|_U \simeq \mathcal{O}_U^n$ we have the following exact sequence:

$$\begin{tikzcd}
\mathcal{F}\,|_U \arrow[r, hook] \arrow[d, "\simeq"] & \mathcal{T}_M\,|_U \arrow[d, "\simeq"] & & \\
\mathcal{O}_U^r \arrow[r, "f"] & \mathcal{O}_U^n \arrow[r] & \mathcal{N}_\mathcal{F}\,|_U \arrow[r] & 0
\end{tikzcd},$$

where f is the homomorphism represented by the matrix of holomorphic functions f_{ij}; from the definition of $\mathrm{Sing}(\mathcal{N}_{\mathcal{F}})$ follows that

$$S(\mathcal{F}) \cap U = \{z \in U \mid \mathrm{rank}(f_{ij}(z)) < p\},$$

where p is the rank of $\mathcal{F}$.

Definition 1.8.8. We call a foliation **regular** if $S(\mathcal{F}) = \emptyset$.

The existence of a regular holomorphic foliation permits us to choose a special atlas for M; this is the statement of the Holomorphic Frobenius Theorem, whose proof can be found e.g. in [31, pages 38–42]. Another good reference on the (real) Frobenius Theorem is the book [23].

Theorem 1.8.9 (Frobenius Theorem)**.** *Let M be a complex manifold, and let $\mathcal{F}$ be a dimension p regular foliation on M. Then there exists an atlas for M, denoted by $\{(U_\alpha, z^1_\alpha, \ldots, z^n_\alpha)\}$, such that on each U_α the generators $v_{1,\alpha}, \ldots, v_{p,\alpha}$ of $\mathcal{F}$ are expressed in coordinates as*

$$v_{i,\alpha} = \frac{\partial}{\partial z^i_\alpha},$$

for $i = 1, \ldots, p$.

Definition 1.8.10. We say that a singular holomorphic foliation $\mathcal{F}$ is **reduced** if for any open set U in M

$$\Gamma(U, \mathcal{T}_M) \cap \Gamma(U \setminus S(\mathcal{F}), \mathcal{F}) = \Gamma(U, \mathcal{F}).$$

We give now an alternative definition of a singular holomorphic foliation using 1-forms, in some sense dual to the one given above using vector fields. The two definitions are equivalent when dealing when reduced foliations; if the definition we are using is not clear from the context we will speak of **foliations of dimension** p when thinking about definition 1.8.5 and **foliations of codimension** $n - p$ when we think about definition 1.8.11.

Definition 1.8.11. (The conormal sheaf of) a **(singular) holomorphic foliation** on M is a coherent subsheaf $\mathcal{G}$ of Ω_M which satisfies the "integrability condition", i.e.,

$$d\,\mathcal{G}_x \subset (\Omega_M \wedge \mathcal{G})_x,$$

for x in $M \setminus S(\mathcal{G})$, where $S(\mathcal{G}) = \mathrm{Sing}(\Omega_M / \mathcal{G})$. We say $\mathcal{G}$ is **regular** on M if $S(\mathcal{G}) = \emptyset$.

Remark 1.8.12. An alternative statement of the Frobenius Theorem is the following. Given a codimension d regular foliation $\mathcal{G}$ such that $S(\mathcal{G}) = \emptyset$ then there exists germs f_i and g_{ij} in $\mathcal{O}_M$, $i, j = 1, \ldots, n-d$ such that $\det(g_{ij})(0) \neq 0$ and that

$$\omega^i = \sum_{j=1}^{n} g^i_j df^j.$$

This means that locally, $\mathcal{G}$ is generated by df^j, for $j = 1, \ldots, n-d$. It is possible to take locally a new chart, in which the last $n - d$ coordinates are given by $f^1, \ldots, f^{n-d}$; in such a chart, we have that the $v \in \mathcal{T}_M$ such that $\omega(v) = 0$ for every ω in $\mathcal{G}$ are generated by $\partial/\partial z^1, \ldots, \partial/\partial z^d$.

Remark 1.8.13. Definition 1.8.5 and Definition 1.8.11 are related as follows. Let $\mathcal{F}$ be a dimension p foliation. We denote by $\mathcal{F}^a$ the annihilator of $\mathcal{F}$:

$$\mathcal{F}^a = \{\omega \in \Omega_M \mid \omega(v) = 0 \text{ for all } v \text{ in } \mathcal{F}\}.$$

It is a possible to check that this is a codimension $n-p$ foliation, refer, e.g., to the exercises of [23]. Given a codimension $n-p$ foliation $\mathcal{G}$ if we denote by $\mathcal{G}^a$, the annihilator of $\mathcal{G}$:

$$\mathcal{G}^a = \{v \in \mathcal{T}_M \mid \omega(v) = 0 \text{ for all } \omega \text{ in } \mathcal{G}\},$$

this is a dimension p reduced foliation.

Remark 1.8.14. There is a canonical way to reduce a non-reduced foliation, refer, e.g., to [7, 28, 31]. Indeed, if $\mathcal{F}$ is a dimension p-foliation, then $\mathcal{G} := \mathcal{F}^a$ is a codimension $n-p$ foliation and we have $S(\mathcal{G}) \subseteq S(\mathcal{F})$. Now, we take $\tilde{\mathcal{F}} = \mathcal{G}^a$; this is a reduced dimension p foliation and $S(\tilde{\mathcal{F}}) = S(\mathcal{G}^a) \subseteq S(\mathcal{G}) \subseteq S(\mathcal{F})$.

In the book by Suwa is stated the following proposition, proved, e.g., in [28].

Proposition 1.8.15. *If a foliation is reduced, then Codim* $S(\mathcal{F}) \geq 2$. *If* $\mathcal{F}$ *is locally free and if Codim* $S(\mathcal{F}) \geq 2$, *then* $\mathcal{F}$ *is reduced.*

In particular, when dealing with a regular foliation of a submanifold S the following definition makes sense.

Definition 1.8.16. Let S be a codimension m complex submanifold of M, a complex n-dimensional manifold. Let $\mathcal{F}$ be a rank l regular foliation of S. We say that an atlas $\{(U_\alpha, z_\alpha^1, \ldots, z_\alpha^n)\}$ is **adapted to S and $\mathcal{F}$** if

- $U_\alpha \cap S = \{z_\alpha^1 = \ldots = z_\alpha^m = 0\}$,
- $\mathcal{F}|_{U_\alpha \cap S}$ is generated by $\partial/\partial z_\alpha^{m+1}|_S, \ldots, \partial/\partial z_\alpha^{m+l}|_S$.

1.9 Splitting of sequences

In this section we will explain a general principle that we are going to use throughout our thesis; this principle was first presented in the paper by Grothendieck [19].

Definition 1.9.1. Let $\mathcal{E}$ and $\mathcal{G}$ be two sheaves of locally free modules over a complex manifold M; an **extension** of $\mathcal{G}$ by $\mathcal{E}$ is a short exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\mathrm{pr}} \mathcal{G} \longrightarrow 0\,. \tag{1.8}$$

Definition 1.9.2. Two extensions of $\mathcal{G}$ by $\mathcal{E}$ are said to be **equivalent** if there exists a morphism φ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_{\mathcal{E}}"] & \mathcal{F} \arrow[r, "\mathrm{pr}"] \arrow[d, "\varphi"] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{G}}"] & 0 \\ 0 \arrow[r] & \mathcal{E} \arrow[r, "\iota'"] & \mathcal{F}' \arrow[r, "\mathrm{pr}'"] & \mathcal{G} \arrow[r] & 0. \end{tikzcd}$$

Definition 1.9.3. We say the sequence (1.8) **splits** if there exists a morphism τ such that $\mathrm{pr} \circ \tau = \mathrm{id}_{\mathcal{G}}$. If the sequence splits we will also say that the extension is **trivial**.

Remark 1.9.4. Please note that a morphism σ splitting $\mathrm{pr} : \mathcal{F} \to \mathcal{G}$ in (1.8) gives rise to another morphism $\tau : \mathcal{F} \to \mathcal{E}$ such that $\tau \circ \iota = \mathrm{id}_{\mathcal{F}}$. This morphism is defined as follows:

$$\tau(\omega) = \iota^{-1}(\omega - \sigma\,\mathrm{pr}(\omega)),$$

for all $\omega \in \mathcal{F}$. Indeed $\omega - \sigma\,\mathrm{pr}(\omega)$ is in the kernel of pr and therefore has a well defined preimage in $\mathcal{F}$.

If we have a map τ splitting $\iota : \mathcal{E} \to \mathcal{F}$ this permits us to define a splitting $\sigma : \mathcal{G} \to \mathcal{F}$ in the following way: if g is an element of $\mathcal{G}$, since pr is surjective we take any of the preimages of g in $\mathcal{F}$ and we denote it by $\tilde{g}$. We define the splitting $\sigma : \mathcal{G} \to \mathcal{F}$ to be

$$\sigma(g) = \tilde{g} - \iota \circ \tau(\tilde{g}).$$

This splitting does not depend on the choice of $\tilde{g}$: if $\tilde{g}_1$ and $\tilde{g}_2$ are two different choices of preimages of g then we have that $\tilde{g}_2 - \tilde{g}_1 = \iota(e)$ for some $e \in \mathcal{E}$ and

$$\tilde{g}_2 - \iota \circ \tau(\tilde{g}_2) - \tilde{g}_1 + \iota \circ \tau(\tilde{g}_1) = \tilde{g}_2 - \tilde{g}_1 - \iota \circ \tau(\tilde{g}_2 - \tilde{g}_1) = 0,$$

since

$$\tilde{g}_2 - \tilde{g}_1 = \iota(e) = \iota(\tau(\iota(e)) = \iota(\tau(\tilde{g}_2 - \tilde{g}_1)).$$

Remark 1.9.5. The extension is called trivial because if the sequence splits we can think of $\mathcal{F}$ as the direct sum of $\mathcal{E}$ and $\mathcal{G}$. Indeed, the direct sum decomposition is given by $\mathcal{F} = \iota(\mathcal{E}) \oplus \sigma(\mathcal{G})$.

Proposition 1.9.6. *The equivalence classes of extensions of $\mathcal{G}$ by $\mathcal{E}$ are in one to one correspondence with the elements of $H^1(M, Hom(\mathcal{G}, \mathcal{E}))$, with the trivial extension corresponding to the zero element.*

We will prove a simplified version of this proposition but first of all we give a receipt to associate to every short exact sequence of locally free modules a cohomology class in $H^1(M, \mathrm{Hom}(\mathcal{G}, \mathcal{E}))$.

Lemma 1.9.7. *Let $\mathcal{E}$ and $\mathcal{G}$ be two sheaves of locally free modules over a complex manifold M; to every extension of $\mathcal{G}$ by $\mathcal{E}$ we can associate a well defined cohomology class in $H^1(M, Hom(\mathcal{G}, \mathcal{E}))$.*

Proof. Let $\mathrm{id}_{\mathcal{G}}$ be the identity homomorphism in $H^0(M, \mathrm{Hom}(\mathcal{G}, \mathcal{G}))$; we take the image of this class in $H^1(M, \mathrm{Hom}(\mathcal{G}, \mathcal{E}))$ through the morphism δ^* defined the Long Exact Sequence Theorem 1.3.9 for Čech cohomology. We compute ω in the following way: let $\{U_\alpha, \mathrm{id}_{\mathcal{G}}\}$ be the identity in $H^0(M, \mathrm{Hom}(\mathcal{G}, \mathcal{G}))$, we take a lift (U_α, τ_α) in $C^0(\mathcal{U}, \mathrm{Hom}(\mathcal{G}, \mathcal{F}))$ and take its Čech coboundary, $\{U_{\alpha\beta}, \tau_\beta - \tau_\alpha\}$. Clearly, $\mathrm{pr} \circ \tau_\beta - \mathrm{pr} \circ \tau_\alpha = 0$, so this is a well defined element of $C^1(\mathcal{U}, \mathrm{Hom}(\mathcal{G}, \mathcal{E}))$. By diagram chasing, as in the proof of Theorem 1.3.9 it is shown this is a Čech cocycle. $\square$

Lemma 1.9.8. *Let ω in $H^1(M, Hom(\mathcal{G}, \mathcal{E}))$ be the class associated to the sequence* (1.8)*. If this class is* 0 *in cohomology then there exists a splitting of the sequence.*

Proof. Suppose we are working with a good cover $\mathcal{U}$ of M so that the Čech cohomology computed with respect to $\mathcal{U}$ is isomorphic to the Čech cohomology of the involved sheaves. Let $\{U_\alpha, \tau_\alpha\}$ be local splittings and compute ω. Suppose ω is 0 in cohomology: this means there exists a cochain $\{U_\alpha, \sigma_\alpha\}$ in $C^0(\mathcal{U}, \mathrm{Hom}(\mathcal{G}, \mathcal{E}))$ whose coboundary is ω, i.e. $\sigma_\beta - \sigma_\alpha = \tau_\beta - \tau_\alpha$. We define now a Čech cochain in $C^0(\mathcal{U}, \mathrm{Hom}(\mathcal{E} \oplus \mathcal{G}, \mathcal{F}))$ as $\{U_\alpha, \theta_\alpha\}$ where θ_α is defined on each U_α as:

$$\theta_\alpha : (v, w) \mapsto (\iota(v - \sigma_\alpha(w)) + \tau_\alpha(w)).$$

We compute now $\delta\{U_\alpha, \theta_\alpha\}$ on each $U_{\alpha\beta}$:

$$\begin{aligned}&\iota(v - \sigma_\beta(w)) + \tau_\beta(w) - \iota(v - \sigma_\alpha(w)) + \tau_\alpha(w)\\ &= \iota(\sigma_\alpha(w) - \sigma_\beta(w)) + \tau_\beta(w) - \tau_\alpha(w) = 0.\end{aligned}$$

Moreover $\mathrm{pr} \circ \theta_\alpha = \mathrm{id}_{\mathcal{G}}$ for each α. So, we have a global isomorphism of sheaves between $\mathcal{E} \oplus \mathcal{G}$ and $\mathcal{F}$ satisfying our requests. $\square$

1.10 Embedding of submanifolds

Remark 1.10.1. In this section we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

In this section we define the important concepts of splitting, k-splitting and comfortable embedding; those notions, who were treated in deep in the series of paper [3],[4],[5] are, on one side, really strong conditions on the embedding while on the other side allow us to treat some really difficult problems; we refer to the cited articles.

We remind some of the notation we are going to use: we shall denote by $\mathcal{O}_M$ the structure sheaf of holomorphic functions on M, by $\mathcal{I}_S$ the ideal sheaf of a subvariety S and by $\mathcal{I}_S^k$ its k-th power as an ideal.

We denote by $\mathcal{T}_M$ and $\mathcal{T}_S$ the tangent sheaves to M and S respectively, where defined. We will denote by Ω_M the sheaf of holomorphic one forms on M.

Let S be a submanifold; we define the **conormal sequence** of sheaves of $\mathcal{O}_S$-modules associated to S as

$$\mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow 0, \tag{1.9}$$

where $d_2 : [f]_2 \mapsto df \otimes [1]_1$ is a well defined map of $\mathcal{O}_S$-modules.

Definition 1.10.2. Let S be a reduced, globally irreducible subvariety of a complex manifold M. We say S **splits** into M if there exists a morphism of sheaves of $\mathcal{O}_S$-modules $\sigma : \Omega_S \to \Omega_{M,S}$ splitting sequence (1.9), i.e., $p \circ \sigma = \mathrm{id}_{\Omega_S}$.

This is only a special case of the concept of splitting sequence from Section 1.9.

Definition 1.10.3. Let S be a reduced, globally irreducible subvariety of a complex manifold M. We put $\mathcal{O}_{S(k)} = \mathcal{O}_M / \mathcal{I}_S^{k+1}$; if f is an element of $\mathcal{O}_M$ we denote its equivalence class in $\mathcal{O}_{S(k)}$ by $[f]_{k+1}$ for any $k \geq 1$. Let $\theta_k : \mathcal{O}_M / \mathcal{I}_S^{k+1} \to \mathcal{O}_M / \mathcal{I}_S$ be the canonical projection given by $\theta_k([f]_{k+1}) = [f]_1$. For $k > h$ we define also canonical projections $\theta_{k,h} : \mathcal{O}_{S(k)} \to \mathcal{O}_{S(h)}$ by $\theta_{k,h}([f]_{k+1}) = [f]_{h+1}$. In general, we write $\Omega_{M,S(k)}$ to denote the sheaf $\Omega_M \otimes \mathcal{O}_{S(k)}$. The **$k$-th infinitesimal neighborhood of S in M** is the ringed space $S(k) := (S, \mathcal{O}_M / \mathcal{I}_S^{k+1})$ together with the canonical inclusion of ringed spaces $\iota_k : S = S(0) \to S(k)$ given by $\iota_k = (\mathrm{id}_S, \theta_k)$. A **$k$-th order lifting** is a splitting morphism $\rho : \mathcal{O}_S \to \mathcal{O}_{S(k)}$ for the exact sequence of rings

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_{S(k)} \longrightarrow \mathcal{O}_S \longrightarrow 0 \,.$$

Definition 1.10.4. Let $\mathcal{O}$, $\mathcal{R}$ be sheaves of rings, $\theta : \mathcal{R} \to \mathcal{O}$ a morphism of sheaves of rings and $\mathcal{M}$ a sheaf of $\mathcal{O}$-modules. A **θ-derivation of $\mathcal{R}$ in $\mathcal{M}$** is a morphism of sheaves of abelian groups $D : \mathcal{R} \to \mathcal{M}$ such that

$$D(r_1 r_2) = \theta(r_1) \cdot D(r_2) + \theta(r_2) \cdot D(r_1)$$

for any $r_1, r_2 \in \mathcal{R}$. In other words, D is a derivation with respect to the $\mathcal{R}$-module structure induced via restriction of scalars by θ.

The splitting condition has a lot of important consequences on the behavior of many objects connected with the submanifold. We shall need the following proposition which stems from a result of commutative algebra [17, Proposition 16.2].

Proposition 1.10.5 ([4] Prop. 2.7)**.** *Let S be a reduced, globally irreducible subvariety of a complex manifold M. Then, there exists is a $1-1$ correspondence among the following classes of morphisms:*

1. *(a) morphisms $\sigma : \Omega_S \to \Omega_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p \circ \sigma = id_{\Omega_S}$;*

   (b) morphisms $\tau : \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$ of sheaves of $\mathcal{O}_S$-modules such that $\tau \circ d_2 = id_{\mathcal{I}_S/\mathcal{I}_S^2}$;

   (c) θ_1-derivations $\tilde{\rho} : \mathcal{O}_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $\tilde{\rho} \circ i_1 = id_{\mathcal{I}_S/\mathcal{I}_S^2}$, where i_1 is the canonical inclusion;

   (d) morphisms $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ of sheaves of rings such that $\theta_1 \circ \rho = id_{\mathcal{O}_S}$.

   Moreover, if any of the former classes is not empty, then there is a $1-1$ correspondence with the following classes of morphisms

2. *(a) morphisms $\tau^* : \mathcal{N}_S \to \mathcal{T}_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p_2 \circ \tau^* = id_{\mathcal{N}_S}$;*

   (b) morphisms $\sigma^* : \mathcal{T}_{M,S} \to \mathcal{T}_S$ of sheaves of $\mathcal{O}_S$-modules such that $\iota \circ \sigma^* = id_{\mathcal{T}_S}$, where $\iota : \mathcal{T}_S \to \mathcal{T}_{M,S}$ is the canonical inclusion.

Proof. We shall prove first that the existence of (1b) implies the existence of (1c). The first thing we remark is that there exists a well defined map $d_2 : \mathcal{O}_S(1) \to \Omega_{M,S}$ defined as $d_2([f]_2) = df \otimes [1]_1$. If there exists a splitting $\tau : \Omega_M \otimes \mathcal{O}_S$ such that $\tau \circ d_2 = \mathrm{id}$ we have that the map $\tau \circ d_2 \circ \iota_1$ is a θ_1-derivation between $\mathcal{O}_{S(1)}$ and $\mathcal{I}/\mathcal{I}_S^2$.

Suppose now we have a θ_1-derivation $\tilde{\rho}$ of $\mathcal{O}_S(1)$ in $\mathcal{I}_S/\mathcal{I}_S^2$; this derivation induces a θ_1-derivation from $\mathcal{O}_M$ to $\mathcal{I}_S/\mathcal{I}_S^2$. If we denote by $\pi : \mathcal{O}_M \to \mathcal{O}_{S(1)}$ the canonical projection the θ_1-derivation is given by $\tilde{\rho} \circ \pi$. So, by the universal property of the module of differentials we have a map $\tilde{\tau} : \Omega_M \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $\tilde{\tau} \circ d = \tilde{\rho} \circ \pi$; if we define $\tau(\omega \otimes [f]_1) = [f]_1 \cdot \tilde{\tau}(\omega)$ we get a morphism of $\mathcal{O}_S$-modules such that $\tau \circ d_2 = \mathrm{id}$.

We want to prove now that we have a 1-to-1 correspondance between the θ_1-derivations and the morphisms of rings between $\mathcal{O}_S$ and $\mathcal{O}_{S(1)}$. Given a θ_1-derivation $\tilde{\rho}$ we define a morphism of rings as $\rho([f]_1) = [f]_2 - \iota_1 \circ \tilde{\rho}([f]_2)$. We want to prove that it is indeed a morphism of rings:

$$\begin{aligned}
\rho([f]_1[g]_1) &= [fg]_2 - \iota_1\tilde{\rho}([fg]_2) \\
&= [f]_2[g]_2 - \iota_1([f]_1\tilde{\rho}([g]_2) + [g]_1\tilde{\rho}([f]_2)) \\
&= ([f]_2 - \iota_1\tilde{\rho}([f]_2))([g]_2 - \iota_1\tilde{\rho}([g]_2)) \\
&= \rho([f]_1)\rho([g]_1),
\end{aligned}$$

since $[\iota_1(\tilde{\rho}([f]_2)\iota_1\tilde{\rho}([g]_2))]_2 = [0]_2$. Conversely, suppose we have a morphism of rings ρ; we define a map $\tilde{\rho}$ as $\tilde{\rho}([f]_2) = \iota_1^{-1}([f]_2 - \rho\theta_1([f]_2))$. We claim this is a θ_1-derivation. Indeed:

$$\begin{aligned}
\iota_1\tilde{\rho}([fg]_2) &= [fg]_2 - \rho\theta_1([fg]_2) \\
&= [fg]_2 - \rho\theta_1([fg]_2) + ([f]_2 - \rho\theta_1([f]_2))([g]_2 - \rho\theta_1([g]_2)) \\
&= [f]_2([g]_2 - \rho([g]_1)) + [g]_2([f]_2 - \rho([f]_1)) \\
&= [f]_2\iota_1\tilde{\rho}([g]_2) + [g]_2\iota_1\tilde{\rho}([f]_2) \\
&= \iota_1(\theta_1([f]_2)\tilde{\rho}([g]_2) + \theta_1([g]_2)\tilde{\rho}([f]_2)),
\end{aligned}$$

where $[([f]_2 - \rho\theta_1([f]_2))([g]_2 - \rho\theta_1([g]_2))]_2 = [0]_2$; the injectivity of ι_1 proves the assertion. We prove now the correspondence between (1b) and (2a). A (nontrivial) consequence of (1b) is that S is non singular [5]; this implies there exists a canonical isomorphism between $(\mathcal{N}_S)^*$ and $\mathcal{I}_S/\mathcal{I}_S^2$; therefore, the dual of the splitting $\tau : \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$ gives rise to a splitting of $\mathrm{pr} : \mathcal{T}_{M,S} \to \mathcal{N}_S$. The last two correspondences, the one between (1b) and (1a) and the one between (2a) and (2b) follow from what we proved above and Remark 1.9.4. $\square$

So, S splits if and only if the sequence

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow \mathcal{O}_{S(1)} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

splits; moreover if S splits, also the sequence

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}_{M,S} \longrightarrow \mathcal{N}_S \longrightarrow 0$$

splits.

Definition 1.10.6. Let $\{(U_\alpha, z_\alpha)\}$ be an atlas adapted to S, complex submanifold of codimension $m \geq 1$ of a complex n-dimensional manifold M. We say that $\mathcal{U}$ is a **splitting atlas** if, for each α and β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$ we have that

$$\left[\frac{\partial z_\beta^p}{\partial z_\alpha^r}\right]_1 = 0.$$

for all $r = 1, \ldots, m, p = m+1, \ldots, n$.

We shall show the connection between the existence of a splitting atlas and the fact that S splits into M after we have given a natural generalization of the concept of splitting, studied in [5] and [4], the notion of k-splitting. In this thesis we will use extensively the notion of 2-splitting.

Definition 1.10.7. Let S be a submanifold of a complex manifold M. We shall say that S **k-splits** into M if and only if there is an infinitesimal retraction of $S(k)$ onto S, that is if there is a $k-$**th order lifting**, i.e., a morphism of sheaves of rings $\rho : \mathcal{O}_S \to \mathcal{O}_M / \mathcal{I}_S^{k+1}$ such that $\theta_k \circ \rho = \mathrm{id}$ or in still other words, if the exact sequence

$$0 \to \mathcal{I}_S/\mathcal{I}_S^{k+1} \hookrightarrow \mathcal{O}_M / \mathcal{I}_S^{k+1} \to \mathcal{O}_M / \mathcal{I}_S \to 0 \tag{1.10}$$

splits as a sequence of sheaves of rings.

We cite now an important proposition; for our thesis only the case of splitting and 2-splitting are necessary and we will give simplified proofs.

Proposition 1.10.8 ([5] Theorem 2.1)**.** *Let S be a codimension m submanifold of a complex manifold M of dimension n. Then S k-splits into M if and only if there exists an atlas $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ adapted to S such that:*

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^k \tag{1.11}$$

for all $r = 1, \ldots, m, p = m+1, \ldots, n$ and for each couple of indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$.

Remark 1.10.9. As a side note, we write down the k-order lifting that comes from the proof in [5] explictly; we have that $\rho_\alpha : \mathcal{O}_S\,|_{U_\alpha} \to \mathcal{O}_{S(k)}\,|_{U_\alpha}$ is

$$\rho_\alpha([f]_1) = \sum_{l=0}^{k} (-1)^l \left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l} \right]_{k+1}.$$

Definition 1.10.10. Let (U_α, z_α) be an atlas adapted to S, complex manifold of codimension $m \geq 1$, of a complex n-dimensional manifold M and let S be k-splitting in M. We say the atlas is **adapted to the k-th order splitting** if

$$\rho([f]_1) = \sum_{l=0}^{k} (-1)^l \left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l} \right]_{k+1}.$$

for each $[f]_1$ in $\mathcal{O}_S(U_\alpha)$ and all indices α such that $U_\alpha \cap S \neq \emptyset$.

Lemma 1.10.11 ([5])**.** *Let S be a codimension m submanifold of a complex manifold M of dimension n. Then S splits into M if and only if there exists an atlas $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ adapted to S such that:*

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S \tag{1.12}$$

for all $r = 1, \ldots, m, p = m+1, \ldots, n$ and for each couple of indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$.

Proof. On each U_α we define

$$\rho_\alpha([f]_1) = [f]_2 - \left[\frac{\partial f}{\partial z_\alpha^r} z_\alpha^r \right]_2,$$

and we compute

$$\begin{aligned}\rho_\beta([f]_1)-\rho_\alpha([f]_1) &= -\left[\frac{\partial f}{\partial z^r_\beta}z^r_\beta\right]_2+\left[\frac{\partial f}{\partial z^r_\alpha}z^r_\alpha\right]_2\\ &= -\left[\frac{\partial f}{\partial z^r_\alpha}\frac{\partial z^r_\alpha}{\partial z^s_\beta}z^s_\beta\right]_2-\left[\frac{\partial f}{\partial z^p_\alpha}\frac{\partial z^p_\alpha}{\partial z^s_\beta}z^s_\beta\right]_2+\left[\frac{\partial f}{\partial z^r_\alpha}z^r_\alpha\right]_2\\ &= -\left[\frac{\partial f}{\partial z^p_\alpha}\frac{\partial z^p_\alpha}{\partial z^s_\beta}z^s_\beta\right]_2.\end{aligned}$$

This class is clearly 0 if the atlas is splitting.

Suppose now S is splitting in M, i.e., there exists a first order lifting ρ; for each α we set $\sigma_\alpha=\rho-\rho_\alpha$. For each α we have that the image of σ_α is contained in $\mathcal{I}_S\,/\,{\mathcal{I}_S}^2$ and is a derivation so we can find $(s_\alpha)^p_r$ such that

$$\sigma_\alpha=(s_\alpha)^p_r[z^r_\alpha]_2\frac{\partial}{\partial z^p_\alpha}.$$

We consider now the change of coordinates

$$\begin{cases}\tilde z^r_\alpha=z^r_\alpha\\ \tilde z^p_\alpha=z^p_\alpha+(s_\alpha)^p_r(z^{m+1}_\alpha,\dots,z^n_\alpha)z^r.\end{cases}$$

From the computations above we have that

$$\begin{aligned}-\left[\frac{\partial z^p_\alpha}{\partial z^s_\beta}\frac{\partial z^s_\beta}{\partial z^r_\alpha}z^r_\alpha\right]_2\frac{\partial}{\partial z^p_\alpha} &= \rho_\beta-\rho_\alpha=\sigma_\beta-\sigma_\alpha\\ &= \left[(s_\beta)^{p'}_{r'}\frac{\partial z^{r'}_\beta}{\partial z^r_\alpha}\frac{\partial z^p_\alpha}{\partial z^{p'}_\beta}-(s_\alpha)^p_r\right]_2[z^r_\alpha]\otimes\frac{\partial}{\partial z^p_\alpha}.\end{aligned}$$

By direct computation, using this last equality, we can verify that the new atlas is such that

$$\frac{\partial\tilde z^p_\alpha}{\partial\tilde z^r_\beta}\in\mathcal{I}_S$$

□

Remark 1.10.12. When we are dealing with S splitting in M, in a splitting atlas all the objects of Proposition 1.10.5 have a nice form in coordinates. Indeed, the following are equivalent

- $\mathcal{U}$ is adapted to ρ;
- for every (U_α,z_α) such that $U_\alpha\cap S\neq\emptyset$ and every f in $\mathcal{O}_{S(1)}|_{U_\alpha}$ one has

$$\tilde\rho([f]_2)=\left[\frac{\partial f}{\partial z^r_\alpha}z^r_\alpha\right]_2;$$

- for every (U_α,z_α) such that $U_\alpha\cap S\neq\emptyset$ and every $r=1,\dots,m$ one has

$$\tau^*(\partial_{r,\alpha})=\frac{\partial}{\partial z^r_\alpha};$$

- for every (U_α, z_α) such that $U_\alpha \cap S \neq \emptyset$ and every $f^k_\alpha \partial/\partial z^k_\alpha \in \mathcal{T}_{M,S}|_{U_\alpha}$ one has
$$\sigma^*(f^k_\alpha \frac{\partial}{\partial z^k_\alpha}) = f^p_\alpha \frac{\partial}{\partial z^p_\alpha}.$$

Lemma 1.10.13 ([5])**.** *Let S be a codimension m submanifold of a complex manifold M of dimension n. Then S 2-splits into M if and only if there exists an atlas $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ adapted to S such that:*
$$\frac{\partial z^p_\beta}{\partial z^r_\alpha} \in \mathcal{I}^2_S \tag{1.13}$$
for all $r = 1, \dots, m, p = m+1, \dots, n$ and for each couple of indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$.

Proof. Suppose first we are in an atlas such that $\partial z^p_\alpha/\partial z^r_\alpha$ is in $\mathcal{I}^2_S$. We define the local 2-order lifting on each U_α as
$$\rho([f]_1) = [f]_3 - \left[\frac{\partial f}{\partial z^r_\alpha} z^r_\alpha\right]_3 + \frac{1}{2}\left[\frac{\partial^2 f}{\partial z^{s_1}_\alpha \partial z^{s_1}_\alpha} z^{s_1}_\alpha z^{s_2}_\alpha\right]_3.$$
Since we have that
$$[z^s_\alpha]_3 = \left[\frac{\partial z^s_\beta}{\partial z^r_\alpha} z^r_\alpha - \frac{1}{2}\frac{\partial^2 z^s_\beta}{\partial z^{r_1}_\alpha \partial z^{r_2}_\alpha} z^{r_1}_\alpha z^{r_2}_\alpha\right]_3, \tag{1.14}$$
for all $s = 1, \dots, m$, it follows that
$$\begin{aligned}
&\rho_\beta([f]_1) - \rho_\alpha([f]_1)\\
&\qquad = \left[\frac{\partial f}{\partial z^s_\beta}\left(\frac{\partial z^s_\beta}{\partial z^r_\alpha} z^r_\alpha - z^s_\beta - \frac{1}{2}\frac{\partial^2 z^s_\beta}{\partial z^{r_1}_\alpha \partial z^{r_2}_\alpha}\right) z^{r_1}_\alpha z^{r_2}_\alpha\right]_3\\
&\qquad\quad - \frac{1}{2}\left[\frac{\partial^2 f}{\partial z^{s_1}_\beta \partial z^{s_2}_\beta}\left(\frac{\partial z^{s_1}_\beta}{\partial z^{r_1}_\alpha}\frac{\partial z^{s_2}_\beta}{\partial z^{r_2}_\alpha} z^{r_1}_\alpha z^{r_2}_\alpha - z^{s_1}_\beta z^{s_2}_\beta\right)\right]_3\\
&\qquad\quad + \left[\frac{\partial f}{\partial z^p_\beta}\frac{\partial z^p_\beta}{\partial z^r_\alpha} z^r_\alpha\right]_3 - \left[\frac{\partial^2 f}{\partial z^s_\beta \partial z^p_\beta}\frac{\partial z^s_\beta}{\partial z^{r_1}_\alpha}\frac{\partial z^p_\beta}{\partial z^{r_2}_\alpha} z^{r_1}_\alpha z^{r_2}_\alpha\right]_3\\
&\qquad\quad - \frac{1}{2}\left[\frac{\partial^2 f}{\partial z^{p_1}_\beta \partial z^{p_2}_\beta}\frac{\partial z^{p_1}_\beta}{\partial z^{r_1}_\alpha}\frac{\partial z^{p_2}_\beta}{\partial z^{r_2}_\alpha} z^{r_1}_\alpha z^{r_2}_\alpha\right]_3\\
&\qquad = 0,
\end{aligned}$$
because of the hypothesis and of equation (1.14).

We prove now the converse; the fact that S is 2-splitting implies that S is splitting and therefore we can suppose that $\partial z^p_\alpha/\partial z^r_\beta$ is in $\mathcal{I}_S$. We denote by ρ the 2-nd order lifting and by $\rho_1 := \theta_{2,1} \circ \rho$ the induced first order lifting. We define ρ_α as above and we set $\sigma_\alpha = \rho - \rho_\alpha$. For each α we have that the image of σ_α is contained in $\mathcal{I}^2_S/\mathcal{I}^3_S$ and it is a derivation, so we can find $(s_\alpha)^p_{r_1 r_2}$ symmetric in the lower indices such that
$$\sigma_\alpha = (s_\alpha)^p_{r_1 r_2}[z^{r_1}_\alpha z^{r_2}_\alpha]_3 \otimes \frac{\partial}{\partial z^p_\alpha}.$$

We consider now the change of coordinates

$$\begin{cases} \tilde{z}^r_\alpha = z^r_\alpha \\ \tilde{z}^p_\alpha = z^p_\alpha + \frac{1}{2}(s_\alpha)^p_{r_1 r_2}(z^{m+1}_\alpha, \dots, z^n_\alpha) z^{r_1}_\alpha z^{r_2}_\alpha. \end{cases}$$

Since we are using a splitting atlas, the computations above tell us that

$$\left[\frac{\partial z^p_\beta}{\partial z^r_\alpha} z^r_\alpha\right]_3 \otimes \frac{\partial}{\partial z^p_\beta} = \sigma_\beta - \sigma_\alpha = \rho_\beta - \rho_\alpha$$
$$= \left[\left(\frac{\partial z^p_\beta}{\partial z^q_\alpha}(s_\alpha)^q_{r_1 r_2} - (s_\beta)^p_{s_1 s_2} \frac{\partial z^{s_1}_\beta}{\partial z^{r_1}_\alpha} \frac{\partial z^{s_2}_\beta}{\partial z^{r_2}_\alpha} z^{r_1}_\alpha z^{r_2}_\alpha\right)\right]_3 \otimes \frac{\partial}{\partial z^p_\beta}.$$

Moreover, since the atlas is splitting we have that $z^p_\beta = \phi_{\beta\alpha}(z^{m+1}_\alpha, \dots, z^n_\alpha) + (h_{\beta\alpha})^p_{r_1 r_2} z^{r_1}_\alpha z^{r_2}_\alpha$ with $(h_{\beta\alpha})^p_{r_1 r_2}$ a holomorphic function on $U_\alpha \cap U_\beta$ symmetric in the lower indices. Therefore, from above, we get

$$\left[2(h_{\beta\alpha})^p_{r_1 r_2} - \frac{\partial z^p_\beta}{\partial z^q_\alpha}(s_\alpha)^q_{r_1 r_2} + (s_\beta)^p_{s_1 s_2} \frac{\partial z^{s_1}_\beta}{\partial z^{r_1}_\alpha} \frac{\partial z^{s_2}_\beta}{\partial z^{r_2}_\alpha}\right]_1 = [0]_1,$$

and hence

$$\left[\frac{\partial z^p_\beta}{\partial z^r_\alpha} - \frac{\partial z^p_\beta}{\partial z^q_\alpha}(s_\alpha)^q_{r_1 r} z^{r_1}_\alpha + (s_\beta)^p_{s_1 s_2} z^{s_1}_\beta \frac{\partial z^{s_2}_\beta}{\partial z^{r_2}_\alpha}\right]_2 = [0]_2.$$

Thanks to this equality we can verify by direct computation that the new atlas is such that

$$\frac{\partial \tilde{z}^p_\alpha}{\partial \tilde{z}^r_\beta} \in \mathcal{I}^2_S.$$

□

Definition 1.10.14. An atlas (U_α, z_α) such that

$$\frac{\partial z^p_\beta}{\partial z^r_\alpha} \in \mathcal{I}^2_S \tag{1.15}$$

for all $r = 1, \dots, m, p = m+1, \dots, n$ and for each couple of indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$ is called a **2-splitting atlas**.

Lemma 1.10.15. *Suppose S is a splitting submanifold of codimension m of a complex n-dimensional manifold M: the sheaf $\mathcal{I}_S/\mathcal{I}^3_S$ has a natural structure of $\mathcal{O}_{S(1)}$-module.*

Proof. Indeed, let f_1 and f_2 be two representatives of $[f]_2$; then $f_1 - f_2$ belongs to $\mathcal{I}^2_S$; let now $[g]$ be a class in $\mathcal{I}_S/\mathcal{I}^3_S$ and $\tilde{g}$ one of its representatives. We define $[f]_2 \cdot [g]_3 = [f_1 \tilde{g}]_3$. This class does not depend on the representative of $[f]_2$ chosen; indeed:

$$[f_1 \tilde{g}]_3 - [f_2 \tilde{g}]_3 = [(f_1 - f_2)\tilde{g}]_3 = [0]_3.$$

This product does not depend on the extension of g chosen, since, given two extensions $\tilde{g}_1$ and $\tilde{g}_2$ we have that $\tilde{g}_2 - \tilde{g}_1$ belongs to $\mathcal{I}^3_S$. □

Now, since S splits into M we have a well defined map $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ which induces a $\mathcal{O}_S$-module structure on $\mathcal{I}_S/\mathcal{I}_S^3$. With this structure the sequence

$$0 \longrightarrow \mathcal{I}_S^2/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow 0 \tag{1.16}$$

is an exact sequence of locally free $\mathcal{O}_S$-modules.

Definition 1.10.16. Let S be an m-codimensional submanifold of a complex manifold M of dimension n. Then S is **comfortably embedded** in M if there exists a first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ such that sequence (1.16) splits as a sequence of $\mathcal{O}_S$-modules.

Lemma 1.10.17 ([5] Theorem 3.5)**.** *Let S be an m-codimensional submanifold of a complex manifold M of dimension n. Then S is comfortably embedded into M if and only if there exists an atlas $U = \{(U_\alpha, z_\alpha)\}$ adapted to S such that*

$$\frac{\partial z_\alpha^p}{\partial z_\beta^r} \in \mathcal{I}_S \quad \text{and} \quad \frac{\partial^2 z_\alpha^r}{\partial z_\beta^{r_1} \partial z_\beta^{r_2}} \in \mathcal{I}_S,$$

for all $r, r_1, r_2 = 1, \dots, m$, $p = m+1, \dots, n$.

In [5] the more general notion of k-comfortable embedding is defined. Even if it is not used in this thesis we define it.

Remark 1.10.18. In [5, Proposition 3.2] it is proved that if S is a complex submanifold of codimension m of a complex manifold M and $\rho : \mathcal{O}_S \to \mathcal{O}_{S(k)}$ is a k-th order lifting, with $k \geq 0$ then for any $1 \leq h \leq k+1$ the lifting ρ induces a structure of locally $\mathcal{O}_S$-free module on $\mathcal{I}_S/\mathcal{I}_S^{h+1}$ so that the sequence

$$0 \longrightarrow \mathcal{I}_S^h/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^h \longrightarrow 0 \tag{1.17}$$

becomes an exact sequence of locally $\mathcal{O}_S$-free modules.

Definition 1.10.19. Let S be a (not necessarily closed) submanifold of a complex manifold M and let $\rho : \mathcal{O}_S \to \mathcal{O}_{S(k)}$ be a k-th order lifting with $k \geq 1$. A **comfortable splitting sequence** ν **associated to** ρ is a $(k+1)$-uple $\nu = (\nu_{0,1}, \dots, \nu_{k,k+1})$ where for $1 \leq h \leq k+1$ each $\nu_{h-1,h} : \mathcal{I}_S/\mathcal{I}_S^h \to \mathcal{I}_S/\mathcal{I}_S^{h+1}$ is a splitting $\mathcal{O}_S$-morphism of the sequence (1.17) with respect to the $\mathcal{O}_S$-module structure induced by ρ. A pair (ρ, ν), where ρ is a k-th order lifting and ν is a comfortable splitting sequence associated to ρ, is called a k**-comfortable** pair for S in M. We say that S is k**-comfortably embedded in** M with respect to ρ if it exists a k-comfortable pair (ρ, ν) for S in M.

The definitions of k-splitting and k-comfortably embedded lead directly to an important notion, the one of k-linearizable manifold.

Definition 1.10.20. Let S be a codimension m submanifold of a n-dimensional complex manifold M. Let $S(k)$ be its k-th infinitesimal neighborhood and $S_N(k)$ be the k-th infinitesimal neighborhood of its embedding as the zero section of its normal bundle in M. We denote by $\mathcal{O}_{N_S}$ the structure sheaf of the normal bundle of S and by $\mathcal{I}_{S,N_S}$ the ideal sheaf of S in N_S. We say $S_N(k)$ is **isomorphic** to $S(k)$ if there exists an isomorphism $\phi : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^{k+1} \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ such that $\theta_k \circ \phi = \theta_k^N$, where $\theta_k : \mathcal{O}_M/\mathcal{I}_S^{k+1} \to \mathcal{O}_S$ and $\theta_k^N : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^{k+1} \to \mathcal{O}_S$ are the canonical projections.

Definition 1.10.21. Let S be a complex submanifold of a complex manifold M. We shall say that S is **k-linearizable** if its k-th infinitesimal neighborhood $S(k)$ in M is isomorphic to its k-th infinitesimal neighborhood $S_N(k)$ in N_S, where we are identifying S with the zero section of N_S.

One of the reasons this notions are important is the next theorem; but first we give a remark.

Remark 1.10.22. In general, given a vector bundle E over a submanifold S, we have that $TE|_S$ is canonically isomorphic to $TS \oplus E$. When E is N_S this implies that the projection on the second summand of $TN_S|_S = TS \oplus N_S$ gives rise to an isomorphism of N_S and N_{0_S}, i.e., the normal bundle of S as the zero section of N_S. Therefore we have an isomorphism between $\mathcal{I}_S/\mathcal{I}_S^2$ and $\mathcal{I}_{S,N_S}/\mathcal{I}_{S,N_S}^2$.

Theorem 1.10.23 ([5], Theorem 4.1)**.** *Let S be a complex submanifold of a complex manifold M, and $k \geq 2$. Then S is k-linearizable if and only if it is k-split and $(k-1)$-comfortably embedded with respect to the k-th order lifting induced by the splitting.*

We shall not use either this definitions or the theorem but they could be used to further develop some of the theory in this thesis, i.e., the extension process in Section 4.4. Indeed in Section 4.4 we use this simpler result.

Proposition 1.10.24 ([4] Prop. 1.3)**.** *Let S be a submanifold of a complex manifold M. Then S splits into M if and only if its first infinitesimal neighborhood $S(1)$ in M is isomorphic to its first infinitesimal neighborhood $S_N(1)$ in N_S, where we are identifying S with the zero section of N_S.*

Proof. If $S_N(1)$ and $S(1)$ are isomorphic there exists an isomorphism of sheaves of rings $\phi : \mathcal{O}_M/\mathcal{I}_{S,N_S}^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ such that $\theta_1 \circ \phi = \theta_1^N$ and we can define a morphism of rings $\rho : \mathcal{O}_S \to \mathcal{O}_S(1)$ by setting $\rho = \phi \circ \rho^N$, since S is splitting in N_S. This morphism of rings satifies all the conditions for the splitting and is therefore a splitting morphism.

Assume S splits in M and let $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ be a first order lifting and let $\tilde{\rho}^N$ be associated θ_1-derivation associated with the splitting of S in N_S. We define a morphism $\phi : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ setting:

$$\phi = \rho \circ \theta_N^1 + i_1 \circ \xi \circ \tilde{\rho}^N,$$

where ξ is the isomorphism in Remark 1.10.22 and $i_1 : \mathcal{I}_S/\mathcal{I}_S^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ is the canonical inclusion. This isomorphism satisfies all the conditions of the claim and so in an isomorphism of sheaves. □

Chapter 2

Localization of Chern classes

2.1 The Bott vanishing theorem

In this section we will prove Bott Vanishing Theorem, following the proof given in [4]. Bott Vanishing Theorem explains the deep link between the existence of a holomorphic connection on a vector bundle and the vanishing of some of its Chern classes.

We recall some of the notation we are going to use; if E be a complex vector bundle on a differentiable manifold M, we denote by $A^p(M, E)$ the set of smooth p-forms with values in E and with $A^p(M)$ the smooth p-forms on M, where A^0 is understood to be the sheaf of smooth functions $A^0(M) = C^\infty(M)$. In the following, we are going to denote by $TM \otimes \mathbb{C}$ the complexified tangent bundle of M, by TM or $T^{(1,0)}M$ its holomorphic part and by $\overline{TM}$ or equivalently $T^{(0,1)}M$ its antiholomorphic part.

Definition 2.1.1. Let M be a complex manifold, let E be a complex vector bundle on M and let F be a subbundle of $T_{\mathbb{C}}M$. We denote by $\rho : T_{\mathbb{C}}M^* \to F^*$ the canonical map, dual to the inclusion $i : F \to T_{\mathbb{C}}M$. A **partial connection for** E **along** F, denoted by (F, δ), is a $\mathbb{C}$-linear application

$$\delta : A^0(M, E) \to A^0(M, F^* \otimes E),$$

which satisfies the "partial" Leibniz rule:

$$\delta(f \cdot s) = \rho(df) \otimes s + f \cdot \delta(s),$$

for $f \in C^\infty(M)$ and $s \in A^0(M, E)$.
A **partial holomorphic connection for** E **along** F is a partial connection such that, if $u \in A^0(M, F)$ and $s \in A^0(M, E)$ are holomorphic then $\delta(s)(u)$ is holomorphic.vanishing theorem, following the proof given in [4]. The Bott Vanishing Theorem explains the deep link between the existence of a holomorphic connection on a vector bundle and the vanishing of some of its Chern classes.

Definition 2.1.2. Let δ be a partial connection for E along F. We say a connection ∇ for E **extends** δ if the following diagram commutes:

$$\begin{tikzcd} A^0(M,E) \arrow[r, "\nabla"] \arrow[d, "\mathrm{Id}"'] & A^0(M, T_{\mathbb{C}}M^* \otimes E) \arrow[d, "\rho\otimes\mathrm{Id}"'] \\ A^0(M,E) \arrow[r, "\delta"] & A^0(M, F^* \otimes E). \end{tikzcd}$$

Lemma 2.1.3. *For each partial connection (F,δ) for E along F there exists a connection ∇ extending it. Moreover, if s is a nonvanishing section in $A^0(M,F)$ such that $\delta(s)=0$, there exists a connection ∇ such that $\nabla(s)=0$.*

Proof. Let $\mathcal{U}=\{U_\alpha\}$ be a covering of M by trivializing neighborhoods. On each U_α, let $e_{\alpha,1},\dots,e_{\alpha,r}$ be a local frame for E on U_α. Then

$$\delta e_{\alpha,i} = \sum_j \gamma^j_{\alpha,i} e_{\alpha,j},$$

with $\gamma^j_{\alpha,i}$ in $A^0(M,F^*)$. On each U_α we can find differential forms $\theta^j_{\alpha,i}$ such that

$$\rho(\theta^j_{\alpha,i}) = \gamma^j_{\alpha,i}.$$

The connection ∇_α having as connection matrix $\theta_\alpha = (\theta^j_{\alpha,i})$ is a connection on U_α extending δ. If we take a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, the connection $\nabla = \sum_\alpha \rho_\alpha \nabla_\alpha$ is a connection for E extending (F,δ).

We prove now the second part of the assertion. Let s be a nonvanishing section in $A^0(M,F)$ such that $\delta(s)=0$. We can choose $e_{1,\alpha} = s|_{U_\alpha}$, and accordingly $\gamma^j_{\alpha,1}=0$ and we can choose $\theta^j_{\alpha,1}=0$ for each $j=1,\dots,r$. $\square$

Definition 2.1.4. Let E be a holomorphic vector bundle. Then we have the differential operator:

$$\bar\partial : A^0(M,E) \to A^0(M, \overline{T^*M} \otimes E).$$

Clearly $(\overline{TM}, \bar\partial)$ is a partial connection for E. A connection ∇ for E extending it is said to be a **connection of type** $(0,1)$.

Remark 2.1.5. If we follow the construction in Lemma 2.1.3, we see that on each U_α, if we take an holomorphic frame $e_{\alpha,1},\dots,e_{\alpha,r}$, we have that $\gamma^j_{\alpha,i}=0$. If the connection is of type $(0,1)$ the map ρ is nothing else that the map which projects a form on its $(0,1)$ component; therefore, the connection matrix θ_α is a matrix of differential forms of type $(1,0)$.

Definition 2.1.6. Let (F,δ) be a partial holomorphic connection. If F is involutive we will say that δ is a **flat partial connection for** E **along** F if

$$\delta(s)([u,v]) = \delta(\delta(s)(v))(u) - \delta(\delta(s)(u))(v),$$

where $u,v\in A^0(M,F)$ and $s\in A^0(M,E)$.

Theorem 2.1.7 (Bott Vanishing Theorem)**.** *Let S be a complex manifold, F a sub-bundle of TS of rank l and E a complex vector bundle on S. Assume we have a partial holomorphic connection on E along F. Then:*

1. *every symmetric polynomial in the Chern classes of E of degree larger than* $\dim S - l + \lfloor l/2 \rfloor$ *vanishes.*

2. *Furthermore, if F is involutive and the partial holomorphic connection is flat then every symmetric polynomial in the Chern classes of E of degree larger than* $\dim S - l$ *vanishes.*

Proof. We can write

$$TS \otimes \mathbb{C} = F \oplus F_1 \oplus T^{(0,1)}S,$$

where F_1 is a C^∞−complement of F in $TS = T^{(1,0)}S$. We define a connection ∇ on E as the direct sum of the given partial holomorphic connection on F, any connection on F_1, and $\bar{\partial}$ on $T^{(0,1)}S$. We want to study the vanishing of the coefficients of ω, the curvature form of ∇. First of all we claim that for each v in the base frame of F and for each $\bar{w}$ in the base frame of $T^{(0,1)}S$

$$\omega(v, \bar{w}) = 0.$$

It is enough to check that the curvature form vanishes when applied to a holomorphic section of E; the holomorphic sections generate $A^0(M, E)$ as a C^∞-module, and the curvature is a tensor (it is linear with respect to C^∞ functions). If σ is a holomorphic section of E we have that

$$\omega(v, \bar{w})\sigma = \nabla_v(\nabla_{\bar{w}}\sigma) - \nabla_{\bar{w}}(\nabla_v\sigma) - \nabla_{[v,\bar{w}]}\sigma = 0.$$

The last equality follows because the three summands are 0. Indeed $\nabla_{\bar{w}} = \bar{\partial}(\sigma)(\bar{w}) = 0$, since σ is holomorphic. For the same reason the second summand vanishes since $\nabla_v\sigma$ is holomorphic because the connection ∇ is a partial holomorphic connection along F; and the last summand is 0 since $[v, \bar{w}] = 0$, due to the integrability of the complex structure.

We claim now that $\omega(\bar{v}, \bar{w}) = 0$ if $\bar{v}, \bar{w} \in T^{(0,1)}S$. Again:

$$\omega(\bar{v}, \bar{w})\sigma = \nabla_{\bar{v}}(\nabla_{\bar{w}}\sigma) - \nabla_{\bar{w}}(\nabla_{\bar{v}}\sigma) - \nabla_{[\bar{v},\bar{w}]}\sigma.$$

The last equality follows from the fact that σ is holomorphic and thus all the three summands are 0, since $[\bar{v}, \bar{w}]$ is in $T^{(0,1)}S$ due to the integrability of the complex structure.

Now, if $n = \dim S$ we can choose local coordinates and local forms $\eta^1, \dots, \eta^n$ such that

$$\{\eta^1, \dots, \eta^n, d\bar{z}^1, \dots, d\bar{z}^n\}$$

is a local frame for the dual of $TS \otimes \mathbb{C}$ respecting the decomposition $TS \otimes \mathbb{C} = F \oplus F_1 \oplus T^{(0,1)}S$; in particular $\{\eta^1|_F, \dots, \eta^l|_F\}$ is a local frame for the dual of F, while $\{\eta^{l+1}|_{F_1}, \dots, \eta^n|_{F_1}\}$ is a local frame for the dual of F_1. Now, what we proved implies that the entries of the curvature matrix in this local frame are composed by forms which are linear combinations of:

$$\eta^i \wedge \eta^j, \eta^i \wedge \eta^p, \eta^p \wedge \eta^q, d\bar{z}^k \wedge \eta^q,$$

where $i, j = 1, \dots, l$, $p, q = l+1, \dots, n$ and $k = 1, \dots, n$.

Furthermore, when F is involutive and ∇ is a flat partial holomorphic connection along F we have that $\omega(v, w) = 0$ for each $v, w \in F$. In this case we have that the curvature matrix has coefficients which are linear combinations of

$$\eta^i \wedge \eta^p, \eta^p \wedge \eta^q, d\bar{z}^k \wedge \eta^q;$$

the product of more of $n-l$ of these forms is 0, which gives us the second part of the assertion. In the non involutive case, the fact that we can take also elements of the type $\eta^i \wedge \eta^j$ implies that the product of $n-l+\lfloor l/2 \rfloor$ of these forms is 0, implying the first part of the assertion. □

Remark 2.1.8. Please note that the Bott Vanishing Theorem in this form not only tells us that the de Rham cohomology class of a Chern class vanishes, but also that the form representing it vanishes.

We refer to [31, pag. 71] for the theory regarding virtual bundles. Before progressing we need a couple of definitions.

Definition 2.1.9. A **virtual bundle** is an element in the K-group $K(M)$ of M [27],[31]. In particular, if we have complex vector bundles E_i with $i = 0, \dots, q$ over a smooth manifold M we may consider the virtual bundle $\xi = \sum_{i=0}^{q}(-1)^i E_i$.

Remark 2.1.10. This definition, apparently obscure, is really useful when we need to define the quotient of a vector bundle that is not defined everywhere. For instance suppose we have two vector bundles E and F, and a map that embeds F as a subbundle of E outside an analytic subset Σ; the virtual bundle $[E-F]$ then coincides with the quotient bundle of E by F outside Σ.

Definition 2.1.11. Let

$$0 \longrightarrow E_q \xrightarrow{\psi_q} \cdots \longrightarrow E_1 \xrightarrow{\psi_1} E_0 \to 0, \tag{2.1}$$

be a sequence of vector bundles on M, and for each $i = 0, \dots, q$, let $\nabla^{(i)}$ be a connection for E_i. We say that the family $\nabla^{(q)}, \dots, \nabla^{(0)}$ is **compatible with the sequence** if, for each i, the following diagram commutes:

$$\begin{tikzcd}
A^0(M, E_i) \arrow[r, "\nabla^{(i)}"] \arrow[d, "\psi_i"'] & A^1(M, E_i) \arrow[d, "1\otimes\psi_i"] \\
A^0(M, E_{i-1}) \arrow[r, "\nabla^{(i-1)}"] & A^1(M, E_{i-1}).
\end{tikzcd}$$

Remark 2.1.12. The following Proposition from the book of Suwa is really general in scope, dealing with virtual bundles and Čech-de Rham cohomology. We explain now the notation, referring to the Sections 1.6 and 1.7 for the background on Chern classes.

We refer to [27, p. 164] for the proof of the formulas which express the Chern classes of $E \oplus F$ as products and sums of the Chern classes of E and F. In general for a direct sum of vector bundles $E \oplus F$, the total Chern class $\mathrm{c}(E \oplus F) = \mathrm{c}(E) \smile \mathrm{c}(F)$. This permits us to compute the Chern classes of $E \oplus F$ as polynomials in the Chern classes of E and F; the simplest example is the first Chern class of the bundle $E \oplus F$: we have that $c_1(E \oplus F) = c_1(E) + c_1(F)$.

A really interesting fact about virtual bundles is that Chern classes behave naturally with respect to them, generalizing the discussion about the Chern class of a direct sum above. In general, if ϕ is a symmetric polynomial, $\xi := \sum_{i=0}^{q} E_i$

is a virtual bundle and $\nabla^\bullet$ denotes a family of connections $\nabla^{(q)}, \ldots, \nabla^{(1)}$ for ξ then we can express $\phi(\xi)$ as a finite sum

$$\phi(\xi) = \sum_l \phi_l^{(0)}(E_0) \wedge \ldots \wedge \phi_l^{(q)}(E_q),$$

where $\phi_l^{(i)}(E_i)$ are polynomials in the Chern classes of E_i for each i and l. Then

$$\phi(\nabla^\bullet) = \sum_l \phi_l^{(0)}(\nabla^{(0)}) \wedge \ldots \wedge \phi_l^{(q)}(\nabla^{(q)})$$

represents the cohomology class of $\phi(\xi)$.

From 1.6.12 we know there exists a form $\phi(\nabla_0^\bullet, \ldots, \nabla_p^\bullet)$, called the Bott difference form such that

$$\sum_{\nu=0}^{p} \phi(\nabla_0^\bullet, \ldots, \widehat{\nabla}_\nu^\bullet, \ldots, \nabla_p^\bullet) + (-1)^p d\phi(\nabla_0^\bullet, \ldots, \nabla_p^\bullet) = 0,$$

where the hat means the hatted family is not taken into consideration.

Proposition 2.1.13 ([31], p. 73)**.** *Suppose sequence* (2.1) *is exact. Let ϕ be a symmetric polynomial and $\nabla_k^\bullet = \{\nabla_k^{(q)}, \ldots, \nabla_k^{(0)}\}$ for $k = 0, \ldots, p$ families of connections compatible with* (2.1) *for the virtual bundle $\tilde{\xi} = \sum_{i=1}^{q}(-1)^{i-1}E_i$. Then:*

$$\phi(\tilde{\nabla}_0^\bullet, \ldots, \tilde{\nabla}_p^\bullet) = \phi(\nabla_0^{(0)}, \ldots, \nabla_p^{(0)})$$

Similarly for other "partitions" of the virtual bundle ξ. In particular

$$\phi(\tilde{\xi}) = \phi(E_0),$$

We state now Bott vanishing theorem in the version for virtual bundles.

Theorem 2.1.14 (Bott Vanishing theorem, [31] pag. 76)**.** *Let M be a complex manifold of dimension n and F an involutive subbundle of rank p of TM. Also, for each $i = 0, \ldots, q$ let E_i be a bundle and let $\nabla_1^{(i)}, \ldots, \nabla_k^{(i)}$ be partial holomorphic connections for E_i along F, then, for any homogeneous symmetric polynomial ϕ of degree $d > n - p$ we have*

$$\phi(\nabla_1^\bullet, \ldots, \nabla_k^\bullet) \equiv 0,$$

where $\nabla_j^\bullet = (\nabla_j^{(q)}, \ldots, \nabla_j^{(0)})$, for $j = 1, \ldots, k$.

2.2 Basic notions about residues

In this section we give a short account of the theory that permits us to compute residues. Even if we postpone a deeper treatment of the Lehmann-Khanedani-Suwa action to the later chapters we will use it in its original form. Thanks to Theorem 2.1.7, an extended version of Bott's vanishing theorem, we know that the existence of a partial holomorphic connection along a subbundle gives rise to the vanishing of some characteristic classes of the bundle endowed with the connection.

We give a sketch of the localization process we use, following [31, p. 194]. Let M be a complex manifold and let S be a compact complex submanifold of dimension n. Suppose we have an involutive coherent subsheaf $\mathcal{F}$ of $\mathcal{T}_M$ of rank l, leaving S invariant, i.e., $\mathcal{F}|_S \subset \mathcal{T}_S$. Define $\Sigma := S \cap S(\mathcal{F})$. We denote by S_0 the set $S \setminus \Sigma$; on S_0 the following short exact sequence of locally free sheaves is defined:

$$0 \longrightarrow \mathcal{F}|_S \longrightarrow \mathcal{T}_{M,S} \xrightarrow{\mathrm{pr}} \mathcal{N}_{\mathcal{F}}|_S \longrightarrow 0\ .$$

We have a flat partial holomorphic connection $(\delta, \mathcal{F}|_S)$ on $\mathcal{N}_{\mathcal{F}}|_S$ along $\mathcal{F}|_S$ (the variation action) on $S \setminus \Sigma$, defined as follow:

$$\delta_v(s) = \mathrm{pr}([\tilde{v}, \tilde{s}]|_S),$$

where $\tilde{v}$ is an extension of v as a section of $\mathcal{F}$ and $\tilde{s}$ is a section of $\mathcal{T}_M$ such that $\mathrm{pr}(\tilde{s}|_S) = s$. Then, Bott Vanishing Theorem implies that the characteristic classes obtained evaluating a symmetric polynomial of degree $n-k$ larger than $n-l$ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$ vanish when restricted to $S \setminus \Sigma$. Let now η be such a characteristic class, obtained from a symmetric polynomial of degree $n-k$; if we inspect the long exact sequence for the cohomology of the pair $(S, S \setminus \Sigma)$

$$\cdots \longrightarrow H^{2(n-k)}(S, S\setminus\Sigma) \longrightarrow H^{2(n-k)}(S) \longrightarrow H^{2(n-k)}(S\setminus\Sigma)\ldots,$$

we can notice that, since η vanishes on $S \setminus \Sigma$ we can lift it to a class in $H^{2(n-k)}(S, S \setminus \Sigma)$; please remark this lift depends on $\mathcal{F}$. Since S is compact we now can apply Poincaré duality P_S and Alexander duality A_Σ obtaining the following commutative diagram:

$$\begin{tikzcd} H^{2(n-k)}(S, S\setminus\Sigma) \arrow[r] \arrow[d, "A_\Sigma"] & H^{2(n-k)}(S) \arrow[d, "P_S"] \\ H_{2k}(\Sigma) \arrow[r, "\iota"] & H_{2k}(S). \end{tikzcd}$$

Since Σ is the union of its connected components Σ_α, its homology is the direct sum of the homologies of each connected component. Thanks to the excision principle we have that, if we take V_α neighborhoods of Σ_α in S such that $V_\alpha \cap V_\beta = \emptyset$ for $\alpha \neq \beta$, the following diagram is commutative:

$$\begin{tikzcd} H^{2(n-k)}(S, S\setminus\Sigma) \arrow[r, "\simeq"] \arrow[d, "A_\Sigma"] & \bigoplus_\alpha H^{2(n-k)}(V_\alpha, V_\alpha \setminus \Sigma_\alpha)\ . \arrow[d, "A_{\Sigma_\alpha}"] \\ H_{2k}(\Sigma) \arrow[r, "\simeq"] & \bigoplus_\alpha H_{2k}(\Sigma_\alpha) \end{tikzcd}$$

Therefore we obtain the following residue formula:

$$\sum_\alpha \iota_\alpha(A_{\Sigma_\alpha})(\eta) = P_S(\eta).$$

In case $k = 0$ what we are doing is to associate to each connected component Σ_α a number and the sum of all these numbers is $P_S(\eta)$. Much of the work in this thesis is devoted to understand how we compute the residue of a characteristic class arising from a polynomial of degree n. In general, the following general principle holds.

Lemma 2.2.1. *Let M be a compact manifold, let E be a vector bundle on M and suppose there exists a partial holomorphic connection ∇ for E along a subbundle F outside Σ, a closed subset. By Bott Vanishing Theorem some of the characteristic classes of E vanish on $M \setminus \Sigma$. Let $\phi(E)$ be such a characteristic class. Then, for each connected component Σ_α of Σ we can define the residue $Res(\nabla, \phi(E); \Sigma_\alpha)$ depending only on the local behaviour of $\phi(E)$ and ∇ near Σ_α.*

2.3 Baum-Bott index theorems

The first theorems that we shall discuss are the Baum-Bott index theorems. The Baum-Bott index theorems refer to localization of Chern classes of the normal bundle to a foliation; we refer to [31]. We deal first with the case when we have a global vector field.

Lemma 2.3.1. *Let M be a complex manifold and suppose there exists a never vanishing holomorphic vector field v. Then there exists a flat partial holomorphic connection for TM along F where F is the subbundle generated by v.*

Proof. We define the partial holomorphic connection in the following way; let s_1, s_2 and s be sections of TM. From the definition of bracket we get:

$$[v, s_1 + s_2] = [v, s_1] + [v, s_2], \quad [v, f \cdot s] = v(f) \cdot s + f \cdot [v, s].$$

Therefore we can define a partial holomorphic connection for TM along F by:

$$\delta(s)(v) = [v, s].$$

Flatness is obvious, since

$$0 = \delta(s)([v, v]) = \delta(\delta(s)(v))(v) - \delta(\delta(s)(v))(v). \tag{2.2}$$

□

Definition 2.3.2. Let v be a holomorphic vector field on a complex manifold M. We will denote by $S(v) := \{p \in M \mid v(p) = 0\}$, the **singularity set of** v. In general we will denote by $\{S_\lambda\}$ the connected components of $S(v)$, so that $S(v) = \bigcup_\lambda S_\lambda$.

From this Lemma, Bott Vanishing Theorem and Lemma 2.2.1 we get the following.

Theorem 2.3.3. *Let M be a compact complex manifold, and let v be a holomorphic vector field with only isolated singularities. Let $S = \bigcup_\lambda p_\lambda$ be the set of such singularities. Let ϕ be a symmetric polynomial of degree n. Then, for each p_λ in $S(v)$ we can define the residue $Res(\phi, v; S_\lambda)$ and the following equation holds true:*

$$\sum_\lambda Res_\phi(v, TM; p_\lambda) = \int_M \phi(TM).$$

However, on complex manifolds, the existence of global holomorphic vector fields is subject to strong topological conditions. The main object of our discussion is a special case of Definition 1.8.4.

Definition 2.3.4. A **dimension one singular holomorphic foliation $\mathcal{F}$ on** M is determined by a system $\{U_\alpha, v_\alpha, f_{\alpha\beta}\}$, where $\{U_\alpha\}$ is an open covering of M, for each U_α we have a vector field v_α and, whenever $U_\alpha \cap U_\beta \neq \emptyset$ there exists a non vanishing holomorphic function $f_{\alpha\beta}$ such that $v_\alpha = f_{\alpha\beta} v_\beta$ on $U_\alpha \cap U_\beta$. We will call the set $S(\mathcal{F}) := \bigcup_\alpha S(v_\alpha)$ the **singular set of the foliation**. The system $\{U_{\alpha\beta}, f_{\alpha\beta}\}$ determines a line bundle which we denote by F and call the **tangent bundle of the foliation**.

Remark 2.3.5. The set $S(\mathcal{F})$ is an analytic set in M, since $S(v_\alpha) \equiv S(v_\beta)$ on $U_\alpha \cap U_\beta$: $v_\beta = f_{\alpha\beta} v_\alpha$ and $f_{\beta\alpha}$ is nonvanishing on $U_\alpha \cap U_\beta$.

Remark 2.3.6. There is a vector bundle homomorphism $\iota : F \to TM$ which sends a section (U_α, f_α) of F to the vector field $v = f_\alpha v_\alpha$. This homomorphism is not injective on $S(\mathcal{F})$. So, on $M_0 = M \setminus S(\mathcal{F})$ the quotient bundle $N_{F_0} = TM_0/F_0$, where $F_0 = F|_{M_0}$ is well defined, and on M_0 we have the following exact sequence:

$$0 \longrightarrow F_0 \xrightarrow{\iota} TM_0 \xrightarrow{\mathrm{pr}} N_{F_0} \longrightarrow 0 \,. \tag{2.3}$$

Definition 2.3.7. We set $\nu_{\mathcal{F}} = [TM - F]$, the **virtual normal bundle** for the foliation $\mathcal{F}$.

Lemma 2.3.8. *Let $\mathcal{F}$ be a dimension one singular holomorphic a foliation on M and let $S(\mathcal{F})$ be its singular set. On $M_0 := M \setminus S(\mathcal{F})$ there exists a flat partial holomorphic connection for $N_{F_0} := TM|_{M_0}/F|_{M_0}$ along $F_0 := F|_{M_0}$.*

Proof. Let v be a section of F_0 and $\mathrm{pr}(w)$ be a section of N_{F_0}. We define the connection as:

$$\delta(\mathrm{pr}(w))(v) = \mathrm{pr}([w, v]).$$

Clearly:

$$\delta(f \cdot \mathrm{pr}(w))(v) = \mathrm{pr}([f \cdot w, v]) = v(f)\,\mathrm{pr}(w) + f \cdot \mathrm{pr}([w, v]),$$

so the Leibniz rule is satisfied. The flatness follows as in (2.2). □

Remark 2.3.9. There exists a natural flat partial holomorphic connection on F_0 along F_0, defined as follows:

$$\delta(v)(w) = [v, w],$$

where v and w are sections of F_0.

Thanks to the Lemma above we see that if ϕ is a symmetric polynomial of degree n we can localize the characteristic class $\phi(\nu_{\mathcal{F}})$ at $S(\mathcal{F})$.

Indeed, let S be a compact connected component of $S(\mathcal{F})$; let U be an open neighborhood of S in M disjoint from the other conponents of $S(\mathcal{F})$. We take a cover $\mathcal{U} = \{U_0, U_1\}$ of U, setting $U_0 = U \setminus S$ and $U_1 = U$. We take families of connections $\nabla_0^\bullet = (\nabla_0^F, \nabla_0^{TM})$ on U_0 and $\nabla_1^\bullet = (\nabla_1^F, \nabla_1^{TM})$. Then $\phi_{\nu_{\mathcal{F}}} = (\phi(\nabla_0^\bullet), \phi(\nabla_1^\bullet), \phi(\nabla_0^\bullet, \nabla_1^\bullet))$. Let ∇ be the flat partial holomorphic connection for N_{F_0} defined in Lemma 2.3.8; we take ∇_0^F as in Remark 2.3.9 and $\nabla_0^{TM} = \nabla \oplus \nabla_0^F$ so that the triple of connections $(\nabla_0^F, \nabla_0^{TM}, \nabla)$ is compatible with the sequence (2.3) (Definition 2.1.11). Every element of this triple of connections is a partial holomorphic connection along F_0 so, thanks to Theorem 2.1.14, we know that $\phi(\nabla_0^\bullet) = 0$ and therefore the cocycle $\phi(\nabla_*^\bullet)$ lies in $A^{2n}(\mathcal{U}, U_0)$ and defines a class in $H^{2n}(U, U \setminus S; \mathbb{C})$.

2.4 Computing the BB index for an isolated singularity

In this section we compute the Baum-Bott index for isolated singularities of a 1-dimensional foliation; we refer to [31].

Suppose we are in the following situation: let M be a complex manifold, let ϕ be a symmetric polynomial of degree n and let v be a vector field vanishing only at isolated singular points $\{p_\lambda\}$, for each point p_λ we can define the residue $\mathrm{Res}_\phi(v, TM; p_\lambda)$ depending only on the local behaviour of v near p_λ. Since it depends only on the local behaviour, we can work in a coordinate neighborhood U of p_λ so that TM is trivial on U and p_λ is the only singular point in U. We can moreover assume that the chart is centered in p_λ, abusing notation, from now on, we will denote p_λ by 0. We write the vector field in coordinates as:

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial z^i}.$$

Let F be the subbundle of TM generated by v outside p_λ and let $\mathcal{V}$ be the cover of U given by $V_0 = U \setminus \{0\}$ and $V_1 = U$. Then, given connections ∇_0 on V_0 and ∇_1 on V_1 the characteristic class we are going to compute is an element of $H^{2n}(A^\bullet(\mathcal{V}))$ represented by $(\phi(\nabla_0), \phi(\nabla_1), \phi(\nabla_0, \nabla_1))$. On V_0 we choose as ∇_0 a connection extending the partial holomorphic connection along F given by Lemma 2.3.8, so, thanks to Bott Vanishing Theorem we have that $\phi(\nabla_0) = 0$. It follows that $(\phi(\nabla_0), \phi(\nabla_1), \phi(\nabla_0, \nabla_1)) = (0, \phi(\nabla_1), \phi(\nabla_0, \nabla_1))$ defines a class in $H^{2n}(A^\bullet(\mathcal{V}), V_0) = H^{2n}(U, U \setminus \{0\}; \mathbb{C})$. Moreover, we can choose the connection ∇_1 to be trivial with respect to $\partial/\partial z^1, \ldots, \partial/\partial z^n$; then, also $\phi(\nabla_1) = 0$. We build a honeycomb cell system adapted to $\mathcal{V}$. Let R_1 be the set

$$R_1 := \{z \in U \mid |v^1(z)| + \ldots + |v^n(z)| \leq n\epsilon^2\},$$

and $R_0 = U_0 \setminus (R_1 \cap U_0)$. Then, the residue we are going to compute is nothing else but (using Alexander duality 1.2.3 and (1.2)):

$$\mathrm{Res}_\phi(TM, v; 0) = \int_{R_{01}} \phi(\nabla_0, \nabla_1).$$

Therefore, the problem consists in computing the Bott difference form $\phi(\nabla_0, \nabla_1)$. The main problem with this computation is that we have to express the connection matrix of ∇_0 in the frame $\partial/\partial z^1, \ldots, \partial/\partial z^n$, but it is difficult to build a nice frame for TM on U_0: the natural choice would be to use the frame

$$v, \frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^n}$$

but, when v^1 is equal to 0 this is not a frame any more. A similar problem arises if we substitute any $\partial/\partial z^i$ with v, there is a problem where v^i vanishes. So, we take the cover $\mathcal{U}$ of V_{01} given by $U_i = \{z \in V_0 \mid v^i(z) \neq 0\}$; let σ be the cochain given by the restriction homomorphism:

$$\sigma = (\phi(\nabla_0, \nabla_1)|_{U_1}, \ldots, \phi(\nabla_0, \nabla_1)|_{U_n}, 0, \ldots, 0).$$

On each U_i we replace $\partial/\partial z^i$ with v to obtain a frame for TM. On each U_i we want to define an ausiliary connection ∇^i, useful for our computations. We give now more and more stringent conditions on ∇^i to ease our computations later. We will compute the connection matrix ∇^i with respect to the frame given by the coordinate fields $\partial/\partial z^1, \ldots, \partial/\partial z^n$. First of all let ∇^i be a $(1,0)$ connection; the second condition is that

$$\nabla^i(\partial/\partial z^j)(\partial/\partial z^k) = 0$$

for $j = 1, \ldots, n$ and $k \neq i$. This means that, if we denote by $(\theta^{(i)})^h_j$ the (j,h) component of the connection matrix, we have that $(\theta^{(i)})^h_j = (a^{(i)})^h_j dz^i$. Now, we ask that $\nabla^i_v w = [v, w]$, which implies that ∇^i is a partial holomorphic connection along v; therefore:

$$\nabla^i_v\left(\frac{\partial}{\partial z^j}\right) = \sum_{h=1}^n (\theta^{(i)})^h_j(v)\frac{\partial}{\partial z^h} = -\sum_{h=1}^n \frac{\partial v^h}{\partial z^j}\frac{\partial}{\partial z^h}$$

and

$$(\theta^{(i)})^h_j(v) = (a^{(i)})^h_j v^i = -\frac{\partial v^h}{\partial z^j}$$

This implies that

$$(\theta^{(i)})^h_j = -\frac{1}{v^i}\frac{\partial v^h}{\partial z^j}dz^i.$$

We define now the $(\theta^{(i)})^h_i$ entry of the connection matrix to be:

$$(\theta^{(i)})^h_i = -\frac{1}{v^i}\frac{\partial v^h}{\partial z^i}dz^i$$

and check that it is a partial holomorphic connection along v:

$$\nabla^i_v\left(\frac{\partial}{\partial z^i}\right) = \sum_{h=1}^n -\frac{1}{v^i}\frac{\partial v^h}{\partial z^i}dz^i\left(\sum_{k=1}^n v^k\frac{\partial}{\partial z^k}\right)\frac{\partial}{\partial z^h} = -\sum_{h=1}^n \frac{\partial v^h}{\partial z^i}\frac{\partial}{\partial z^h},$$

as desired.

Now we shall find a cochain θ such that $\sigma - \theta = D\eta$, and which contains only Chern classes involving in their computation ∇_1 and the ∇^i's, the holomorphic partial connections along F defined on each U_i. We will proceed playing Tic-Tac-Toe (for a reference about this term [11]). The idea is to find a cochain ω_1 in $K^{0,2n-2}$ such that $d\omega_1 = \sigma$ and then compute the cochain $\sigma - D\omega_1$. This cochain, as opposite to the procedure we used in proving the isomorphism between Čech-de Rham and de Rham cohomologies, has now no components in $K^{0,2n-1}$. The idea is to iterate this process finding cochains $\omega_1, \ldots, \omega_{n-1}$ such that $\theta = \sigma - D(\omega_1 + \ldots + \omega_{n-1})$ has only $K^{n-1,n}$ components; then, clearly $\eta = \omega_1 + \ldots + \omega_n$. As a matter of fact, the special structure of our covering gives us a lot of help: indeed, for each i let $\phi(\nabla_0, \nabla_1, \nabla^i)$ be the Bott difference form for $\nabla_0, \nabla_1, \nabla^i$. By the very definition of the Bott difference form we have:

$$d\phi(\nabla_0, \nabla_1, \nabla^i) = -\phi(\nabla_1, \nabla^i) + \phi(\nabla_0, \nabla^i) - \phi(\nabla_0, \nabla_1).$$

Thanks to the Bott vanishing theorem we have that $\phi(\nabla_0, \nabla^i) = 0$ since both are partial holomorphic connections along F; moreover ∇_1, ∇^i are partial holomorphic connection along $TM_i := \mathrm{Span}\{\partial/\partial z^1, \ldots, \widehat{\partial/\partial z^i}, \ldots, \partial/\partial z^n\}$, where with the hat we mean we are not taking that coordinate field into consideration. Then, by Bott Vanishing Theorem, we know that $\phi(\nabla_1, \nabla^i) = 0$. Therefore:

$$d\phi(\nabla_0, \nabla_1, \nabla^i) = -\phi(\nabla_0, \nabla_1)$$

and we can take

$$\omega_1 = (-\phi(\nabla_0, \nabla_1, \nabla^1), \dots, -\phi(\nabla_0, \nabla_1, \nabla^n), 0, \dots, 0).$$

We have that $(\sigma - D\omega_1)_i = 0$ and

$$(\sigma - D\omega_1)_{ij} = -\phi(\nabla_0, \nabla_1, \nabla^j) + \phi(\nabla_0, \nabla_1, \nabla^i).$$

Again from the definition of Bott difference form we have:

$$\begin{aligned} d\phi(\nabla_0, \nabla_1, \nabla^i, \nabla^j) = &- \phi(\nabla_1, \nabla^i, \nabla^j) + \phi(\nabla_0, \nabla^i, \nabla^j) \\ &- \phi(\nabla_0, \nabla_1, \nabla^i) + \phi(\nabla_0, \nabla_1, \nabla^j). \end{aligned}$$

Again, since $\nabla_0, \nabla^i, \nabla^j$ are partial holomorphic connections along F we have $\phi(\nabla_0, \nabla^i, \nabla^j) = 0$ and since $\nabla_1, \nabla^i, \nabla^j$ are partial holomorphic connections along $TM_i \cap TM_j$ we have that $\phi(\nabla_1, \nabla^i, \nabla^j) = 0$. So:

$$(\sigma - D\omega_1)_{ij} = d\phi(\nabla_0, \nabla_1, \nabla^i, \nabla^j).$$

In our situation we have that in general, for $k = 1, \dots, n-1$, we have:

$$\begin{aligned} (-1)^{k+3} d\phi(\nabla_0, \nabla_1, \nabla^{i_1}, \dots, \nabla^{i_k}) = &\phi(\nabla_1, \nabla^{i_1}, \dots, \nabla^{i_k}) \\ &- \phi(\nabla_0, \nabla^{i_1}, \dots, \nabla^{i_k}) \\ &+ \sum_{\nu=2}^{k+2} \phi(\nabla_0, \nabla_1, \nabla^{i_1}, \dots, \widehat{\nabla^{i_{\nu-2}}}, \dots, \nabla^{i_k}). \end{aligned}$$

For $k = 1, \dots, n$ we have that $\phi(\nabla_0, \nabla^{i_1}, \dots, \nabla^{i_k}) = 0$ since the connections are all partial holomorphic connections along F. For $k = 1, \dots, n-2$ we have that $\phi(\nabla_1, \nabla^{i_1}, \dots, \nabla^{i_k}) = 0$ since the connections are all partial holomorphic connections along $\bigcap_{\nu=1}^k TM_{i_\nu}$. We can iterate the process and we obtain finally a cochain which has only the component in $K^{n-1,n}$; this cochain has components $\theta = (0, \dots, 0, -\phi(\nabla_1, \nabla^1, \dots, \nabla^n))$.

Remark 2.4.1. In the book by Suwa [31, p. 105], the cochain η such that $\sigma - \theta = D\eta$ is built: in components the cochain is given as $(\eta)_{i_0,\dots,i_k} = \phi(\nabla_0, \nabla_1, \nabla^{i_1}, \dots, \nabla^{i_k})$. Following the same line of thoughts we followed above it is possible to compute that:

$$\begin{cases} (D\eta)_i = -\phi(\nabla_0, \nabla_1) \\ (D\eta)_{i_0 \dots i_k} = 0 \quad \text{for k=1,...,n-2} \\ (D\eta)_{1 \dots n} = -\phi(\nabla_1, \nabla^1, \dots, \nabla^n). \end{cases}$$

Let now ι be the inclusion map $\partial R_1 \hookrightarrow U_0$; we denote by $\iota^* \mathcal{U}$ the covering of ∂R_1 by the open sets $\partial R_1 \cap U^i$. As a system of honeycomb cells adapted to $\iota^* \mathcal{U}$ we take

$$R^i = \{z \in \partial R_1 \mid |v^i(z)| \geq |v^j(z)| \text{ for } j \neq i\},$$

and for $(i_0, \dots, i_k)$ with $1 \le i_0 < \dots < i_k \le n$ we set $R^{i_0 \dots i_k} = \bigcap_{\nu=0}^k R_{i_\nu}$. We can now consider the integration:

$$\int_{\partial R_1} : A^{2n-1}(\iota^* \mathcal{U}) \to \mathbb{C}.$$

Since σ and θ represent the same class in cohomology we have that

$$\int_{\partial R_1} \sigma = \int_{\partial R_1} \theta$$

and then:

$$\begin{aligned}\int_{R_{01}} \phi(\nabla_0, \nabla_1) &= -\int_{\partial R_1} \phi(\nabla_0, \nabla_1) = -\sum_{i=1}^n \int_{R^i} \phi(\nabla_0, \nabla_1) \\ &= \int_{R^{1\dots n}} \phi(\nabla_1, \nabla^1, \dots, \nabla^n).\end{aligned}$$

We want now to compute $\phi(\nabla_1, \nabla^1, \dots, \nabla^n)$ explictly. Hence, we take the connection $\tilde{\nabla} = (1 - \sum_{i=1}^n t_i)\nabla_1 + \sum_{i=1}^n t_i \nabla^i$; its connection matrix is given by

$$\tilde{\theta} = (1 - \sum_{i=1}^n t_i)\theta_1 + \sum_{i=1}^n t_i \theta^i.$$

We defined θ_1 to be trivial with respect to the frame of the coordinate fields; then the curvature form $\tilde{\omega}$ of $\tilde{\nabla}$ is given by

$$\tilde{\omega} = -A \otimes \sum_{i=1}^n dt^i \wedge \frac{dz_i}{v^i} + \text{terms not containing } dt^i,$$

where A is the matrix having as components $(A)^i_j = \partial v^i / \partial z^j$. Using the fact that $\int_{\Delta^n} dt^1 \wedge \dots \wedge dt^n = 1/n!$ we get, by the definition of Bott difference form

$$\phi(\nabla_1, \nabla^1, \dots, \nabla^n) = (-1)^{[\frac{n}{2}]} \frac{\phi(-A) dz^1 \wedge \dots \wedge dz^n}{v^1 \dots v^n}, \tag{2.4}$$

and taking into account the orientation of $R^{1,\dots,n}$ we get that

$$\mathrm{Res}_\phi(v, 0) = \int_{\partial R^1} \frac{\phi(-A) dz^1 \wedge \dots \wedge dz^n}{v^1 \dots v^n}.$$

Definition 2.4.2. Let $\mathcal{O}_0^n$ denote the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n$ and $v^1, \dots, v^n$ germs in $\mathcal{O}_0^n$ such that $\{z \in \mathbb{C}^n \mid v^1(z) = \dots = v^n(z) = 0\} = \{0\}$. For a germ ω at 0 of holomorphic n-form we choose a neighborhood U of 0, where $v^1, \dots, v^n$ and ω have representatives and we let Γ be the n-cycle in U defined by

$$\Gamma = \{z \in U \mid |v^1(z)| = \dots = |v^n(z)| = \epsilon\},$$

where $\epsilon > 0$ is small. We orient Γ so that the form $d\theta_1 \wedge \dots \wedge d\theta_n$ is positive, $\theta_i = \arg(v^i)$. Then we set

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ v^1, \dots, v^n \end{bmatrix} = \frac{1}{(2\pi\sqrt{-1})^n} \int_\Gamma \frac{\omega}{v^1 \cdots v^n},$$

and call it the **Grothendieck residue of** ω **at** 0 **with respect to** $v^1, \dots, v^n$.

Let ϕ be a degree n symmetric polynomial, v be a holomorphic vector field with an isolated singularity at 0 and A be the matrix $(A)^i_j = \partial v^i/\partial z^j$. Then

$$\mathrm{Res}_\phi(v,0) = \left(\frac{2\pi}{\sqrt{-1}}\right)^n \mathrm{Res}_0 \begin{bmatrix} \phi(A)dz^1 \wedge \ldots \wedge dz^n \\ v^1, \ldots, v^n \end{bmatrix}.$$

In the introduction of [7] Baum and Bott explain an algorithm developed by Hartshorne to compute the residue

$$\mathrm{Res}_0 \begin{bmatrix} \phi(A)dz^1 \wedge \ldots \wedge dz^n \\ v^1, \ldots, v^n \end{bmatrix}.$$

Since the origin is an isolated 0 of the v^i's, then, by Hilbert Nullstellensatz, there exists $k_1, \ldots, k_n$ such that, for each i, we have that $(z^i)^{k_i}$ belongs to the ideal generated by $v^1, \ldots, v^n$. Hence there exist holomorphic functions b^i_j such that

$$(z^i)^{k_i} = \sum_{j=1}^n b^i_j v^j.$$

If we denote by B the matrix with components $(B)^i_j := b^i_j$ then:

$$\mathrm{Res}_0 \begin{bmatrix} \phi(A)dz^1 \wedge \ldots \wedge dz^n \\ v^1, \ldots, v^n \end{bmatrix} = \mathrm{Res}_0 \begin{bmatrix} \phi(A)\det(B)dz^1 \wedge \ldots \wedge dz^n \\ (z^1)^{k_1}, \ldots, (z^n)^{k_n} \end{bmatrix}.$$

By the Cauchy integral formula, the residue is the coefficient of

$$dz^1 \wedge \ldots \wedge dz^n/(z^1 \cdots z^n)$$

in the Laurent series for

$$\frac{\phi(A)\det(B)dz^1 \wedge \ldots \wedge dz^n}{(z^1)^{k_1} \cdots (z^n)^{k_n}}.$$

Chapter 3

Existence of partial holomorphic connections

Remark 3.0.3. In this chapter we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

3.1 Splitting of the Atiyah sequence and existence of partial holomorphic connections

In this section we give a proof of a result similar to the one in [6] about the existence of partial holomorphic connections. This result, about partial holomorphic connections, was first stated in [4]; we give a complete proof (whilst almost obvious) for the sake of completeness.

The first thing we remind is the result of Grothendieck in [19] we cited in Section 1.9 that tells us that there exists a bijective correspondence between equivalence classes of extensions of a coherent sheaf $\mathcal{E}$ by a coherent sheaf $\mathcal{G}$, i.e., the equivalence classes of short exact sequences

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

and the elements of $H^1(M, \mathrm{Hom}(\mathcal{G}, \mathcal{E})) = \mathrm{Ext}^1_{\mathcal{O}_M}(\mathcal{G}, \mathcal{E})$. In particular, the sequence splits if and only if the associated cohomology class vanishes. The result in the paper of Atiyah shows how the existence of a holomorphic connection on a vector bundle E is equivalent to the splitting of the sequence

$$0 \to \mathrm{Hom}(E, E) \to \mathcal{A}_E \to \mathcal{T}_M \to 0 \tag{3.1}$$

where $\mathcal{A}_E$ is the Atiyah sheaf of E, an important object that we will define in this section.

Definition 3.1.1. The cohomological obstruction to the splitting of sequence (3.1) is called the **Atiyah class** of E.

We want to prove a similar result for partial holomorphic connections: the existence of a partial holomorphic connection (F, δ) is equivalent to the splitting of the sequence:

$$0 \to \mathrm{Hom}(E, E) \to \mathcal{A}_{E,\mathcal{F}} \to \mathcal{F} \to 0, \tag{3.2}$$

the restriction of sequence (3.1) to a subbundle of TM.

Lemma 3.1.2. *Let E be a vector bundle over M. Then there exists a canonical short exact sequence of vector bundles:*

$$0 \longrightarrow Hom(E,E) \longrightarrow A_E \xrightarrow{pr} TM \longrightarrow 0$$

where TM is the tangent bundle, $Hom(E,E)$ is the endomorphism bundle of E and A_E is defined in the proof. Moreover, if F is any subbundle of $\mathcal{T}_M$, there exists a short exact sequence of vector bundles

$$0 \to Hom(E,E) \to A_{E,F} \to F \to 0,$$

where $A_{E,F}$ is $pr^{-1}(F)$.

Proof. Let E be the vector bundle of rank r and let $\pi : P \to M$ be the associated principal bundle (refer to [30] for a complete treatment), i.e., the bundle of r-frames of E, with structure group G. We denote by TP the tangent bundle of P; if G acts on P it operates also on TP. We define $A_E := TP/G$; every point of A_E is a vector field tangent to P along one of its fibers, invariant under G. Let q be a point in A_E. If we choose U so small that it is a trivializing subset for P we can denote q by a triple $[x, v, B]$; this class represents all the points in $TP|_{\pi^{-1}(x)}$ of the form $(x, A, v, A \cdot B)$, where A is an element in G and B is an element in the Lie algebra of the group G. Please note that these are the equivalence classes fibrewise, i.e., for a fixed x; the quotient structure of TP/G is more complicated. We make clear what does it mean to quotient TP by the action of G on U. Indeed, let γ be curve in P, such that $\gamma(0) = (x, A)$ and $\dot\gamma = (v, B)$. Let g be a holomorphic map $U \to G$, where G is the structure group of P. Then, differentiating, we see that

$$g \cdot \gamma = (\gamma(t), g(\gamma(t))A(t))$$

and the action on $TP_{(x,A)}$ is

$$(x, A, v, B) \mapsto (x, g(x)A, v, dg(x)(v) \cdot A + g(x) \cdot B), \qquad (3.3)$$

where by dg we denote the matrix having as entries the differentials of the entries of g and by $dg(v)$ we denote the matrix having as entries the evaluation of those differentials on v.

Since U is a trivializing subset there exists a section s of P on U; if we restrict TP to $s(U)$ we see there is a one to one correspondence between $A_E|_U$ and $TP|_{s(U)}$, given by

$$q = [x, v, B] \mapsto (x, s(x), v, s(x) \cdot B),$$

this permits us to define a vector bundle structure on A_E induced by the one on $TP|_{s(U)}$. We have to prove this vector bundle structure is well defined; indeed if s' is another cross-section there exists a holomorphic map $g : U \to G$ such that $s'(x) = g(x) \cdot s(x)$ and therefore an isomorphism of vector bundles $TP|_{s(U)} \simeq TP|_{s'(U)}$ and a commutative diagram:

$$\begin{tikzcd} A_E|_U \arrow[r] \arrow[dr] & TP|_{s(U)} \arrow[d] \\ & TP|_{s'(U)}, \end{tikzcd}$$

implying that the structure of vector bundle is well defined.

Since P is a principal bundle and on principal bundles the structure group acts fiberwise we have now that $d\pi : TP \to TM$ factors through a map $\mathrm{pr} : A_E \to TM$. We denote now by V the subbundle $\ker(d\pi)$ of TP; we have that G operates on V as well. Put $R = V/G$; reasoning as above we can prove that R is a vector subbundle of A_E. A point in R is nothing else but an equivalence class $[x, 0, B]$, representing all the points in TP of the form $(x, A, 0, A \cdot B)$, those are indeed the left invariant vector fields along the fibers of P and since the space of left invariant vector fields on a Lie Group G is isomorphic to the Lie Algebra of the group $\mathfrak{g}$ we have an isomorphism $V \simeq P \times \mathfrak{g}$. Now, the action of G on V corresponds in this isomorphism to the adjoint action. We consider $R \simeq P \times_G L$, where $P \times_G L$ is the quotient of $P \times L$ with respect to the adjoint action; this is a bundle of Lie Algebras on M, with fibre isomorphic to $\mathfrak{g}$. In general, given a vector space with automorphism group G, its endomorphism group is isomorphic to $\mathfrak{g}$. Hence $R \cong \mathrm{End}(E) = \mathrm{Hom}(E, E)$.

Now, we restrict sequence (3.1) to F; clearly, if we take $A_{E,F} = \mathrm{pr}^{-1}(F)$ we have surjectivity in the second entry of sequence (3.2); now, the kernel of pr is nothing else that $\mathrm{Hom}(E, E)$, as above, and this proves the second assertion. □

Remark 3.1.3. Remark that a short exact sequence of vector bundles gives rise to a short exact sequence for their sheaves of sections. Therefore the preceeding proposition proves the existence of sequence (3.1) and (3.2).

We prove now the main result of this section.

Proposition 3.1.4. *Let E be a holomorphic vector bundle on a complex manifold M. Then E admits a partial holomorphic connection (F, δ) if and only if sequence*

$$0 \to \mathcal{H}om(E, E) \to \mathcal{A}_{E,F} \to F \to 0,$$

splits, or equivalently if the cohomology class which representes the obstruction to the splitting in $H^1(M, \mathcal{H}om(F, \mathcal{H}om(E, E)))$ vanishes.

Proof. The notation in this proof refers to the notation in Lemma 3.1.2. Since $\pi : P \to M$ is a principal bundle there exists a cover of M by trivializing neighborhoods; if U is any of this trivializing neighborhoods $TP|_U$ can be seen as the direct sum of $V|_U = \ker(d\pi)$ and $TM|_U = \mathrm{Im}(d\pi)$. This isomorphism commutes with the action of G and gives rise to an isomorphism

$$TP/G|_U = A_E|_U \cong V/G|_U \oplus TM|_U.$$

If we are working with the vector bundle $A_{E,F}$ this gives rise to an isomorphism $A_{E,F}|_U \cong V/G|_U \oplus F|_U = \mathrm{Hom}(E, E)|_U \oplus F|_U$. While this isomorphism is defined for each U_α in the cover $\{U_\alpha\}$ of M there is an obstruction for the local isomorphisms to glue together and give rise to a global isomorphism; this obstruction is the cohomology class associated to the sequence. Indeed, if we denote by $\tau_\alpha : V/G|_{U_\alpha} \oplus F|_{U_\alpha} \to A_{E,F}|_{U_\alpha}$ the local isomorphisms, we can define local lifts of the identity homomorphism on TM by $\sigma_\alpha(t) = \tau_\alpha(0 \oplus t)$. Now, the difference between two such isomorphisms is exactly the obstruction we are looking for; we will denote it by ω and $\omega_{\alpha\beta} = \sigma_\beta - \sigma_\alpha$. We compute $\tau_\beta^{-1}(\omega)_{\alpha\beta}$. Indeed:

$$\tau_\beta^{-1}(\omega_{\alpha\beta})(0 \oplus t) = (0 \oplus t) - \tau_\beta^{-1}\tau_\alpha(0 \oplus t).$$

We want to understand now what is $\tau_\beta^{-1}\tau_\alpha(0\oplus t)$; to do that, we remind ourselves that we defined τ_α as the isomorphism induced on $A_E|_{U_\alpha}$ by the isomorphisms of $TP|_{U_\alpha}$. Now, the transition functions for $TP|_{U_\alpha}$ are given, if we let G act on the left on P, we have that $g_{\alpha\beta}(x)$, the transition maps for P, induce homomorphisms $dg_{\alpha\beta}(x)$ from TM (and therefore F) to $\mathrm{Hom}(E,E)$. Hence

$$\tau_\beta^{-1}(\omega_{\alpha\beta})(0\oplus t) = (0\oplus t) - (dg_{\alpha\beta}(t)\oplus t),$$

and ω is represented by $\omega_{\alpha\beta} = \tau_\beta dg_{\alpha\beta}$. If ω vanishes in cohomology then there exists a cochain $\{U_\alpha, \eta_\alpha\}$ in $C^0(\mathrm{Hom}(F,\mathrm{Hom}(E,E)))$ such that $\eta_\beta - \eta_\alpha = \omega_{\alpha\beta}$. If now we define $\theta_\alpha = \tau_\alpha^{-1}\eta_\alpha$ then

$$\theta_\beta = dg_{\alpha\beta} + g_{\alpha\beta}\theta_\alpha g_{\alpha\beta}^{-1}.$$

The matrices of 1-forms θ_α are exactly the connection matrices for a partial holomorphic connection.

We prove the converse, i.e., the existence of a partial holomorphic connection (F,δ) implies the splitting of sequence (3.2). Let $\{U_\alpha\}$ be a collection of trivializing neighborhoods of E. If $\{e_{1,\alpha},\dots,e_{n,\alpha}\}$ is a frame for E on U_α it gives rise to a section $A_\alpha(x)$ of P on U_α. If $v(x)$ is a section of F over U_α we have that

$$\delta_v e_{j,\alpha} = \sum_{i=1}^n b^i_{j,\alpha}(x) e_{i,\alpha}.$$

Let $B_\alpha : U_\alpha \to \mathfrak{g}$ be the holomorphic map defined by $(B_\alpha)^i_j(x) = b^i_{\alpha,j}(x)$.

We define a local holomorphic map $\tilde\tau_\alpha$ which sends an holomorphic section of F over U_α to a holomorphic section of TP over U_α by setting

$$\tilde\tau_\alpha^{A_\alpha}(v) = (x, A_\alpha(x), v, B_\alpha(x)).$$

We claim this map induces a local splitting $\tau_\alpha : \mathcal{F}(U_\alpha) \to \mathcal{A}_E(U_\alpha)$, where by $\mathcal{F}$ and $\mathcal{A}_E$ we denote the sheaf of holomorphic sections of F and A_E respectively, by taking equivalence classes

$$\tau_\alpha(v) = [x, A_\alpha(x), v, B_\alpha(x)].$$

Indeed, if we choose another frame $A'_\alpha(x)$ we have that there exists a holomorphic map $g : U_\alpha \to G$ such that $A'_\alpha(x) = g(x)\cdot A_\alpha(x)$. We denote by $B'_\alpha(x)$ the analogous of $B_\alpha(x)$ for the frame $A'_\alpha(x)$. Since (F,δ) is a partial holomorphic connection

$$\delta_v(e'_{i,\alpha}) = \delta_v(\sum_{j=1}^n g^j_i e_{j,\alpha}) = \sum_{j=1}^n (dg^j_i(v)e_{j,\alpha} + (g_{\beta\alpha})^j_i \delta_v(e_{j,\alpha}))$$

it follows

$$B'_\alpha = dg(x)(v)A_\alpha(x) + g(x)B_\alpha(x),$$

which implies that

$$[x, A'_\alpha(x), v, B'_\alpha(x)] = [x, g(x)A_\alpha(x), v(x), dg(x)(v(x))A_\alpha(x) + g(x)B_\alpha(x)]$$

and, since we are quotienting by the action of G we have that

$$[x, g(x)A_\alpha(x), v(x), dg(x)(v(x))A_\alpha(x) + g(x)B_\alpha(x)] = [x, A_\alpha(x), v(x), B_\alpha(x)]$$

and the local splitting is well defined.

Let now U_β be another trivializing neighborhood for E, such that $U_\alpha \cap U_\beta \neq \emptyset$. We want to check that, if v is a holomorphic section of F over $U_\alpha \cap U_\beta$ then $\tau_\alpha(v) = \tau_\beta(v)$; let A_α be a frame for E on U_α and A_β a frame for E on U_β; we have that

$$\tau_\beta(v(x)) = [x, A_\beta(x), v(x), B_\beta(x)]$$

but, since (F, δ) is a partial holomorphic connection

$$B_\beta(x) = dg_{\beta\alpha}(x)(v)A_\alpha(x) + g_{\beta\alpha}(x)B_\alpha(x),$$

and so

$$\begin{aligned}\tau_\beta(v(x)) =&[x, A_\beta(x), v(x), B_\beta(x)]\\ =&[x, g_{\beta\alpha}(x)A_\alpha(x), v(x), dg_{\beta\alpha}(x)(v(x))A_\alpha(x) + g_{\beta\alpha}(x)B_\alpha(x)]\\ =&[x, A_\alpha(x), v(x), B_\alpha(x)] = \tau_\alpha(v(x)).\end{aligned}$$

So, the cochain $\{U_\alpha, \tau_\alpha\}$ is a Čech cocycle and gives rise to a global splitting of the sequence. □

3.2 The Atiyah class for the normal bundle of a foliation in the ambient tangent bundle

In this section we shall compute the Atiyah class of $\mathcal{N}_{\mathcal{F},M}$ with respect to $\mathcal{F}$. This sheaf of sections is defined to be an analogous of the restriction of the normal sheaf to the foliation in the ambient tanget bundle for foliations of S.

We recall some of the notation used; if M is a complex manifold and S a complex (regular) submanifold, let $\mathcal{T}_M$ be the sheaf of sections of the holomorphic tangent bundle of M; if we denote by $\mathcal{O}_M$ the regular functions on M and by $\mathcal{O}_S$ the regular functions on S we define $\mathcal{T}_{M,S} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$.

Definition 3.2.1. Let S be a submanifold of a complex manifold M and let $\mathcal{F}$ be a foliation of S. We think of $\mathcal{F}$ as a subsheaf of $\mathcal{T}_{M,S}$, and we define the **normal bundle to the foliation in the ambient tangent bundle** as the quotient of $\mathcal{T}_{M,S}$ by $\mathcal{F}$ and we will denote it by $\mathcal{N}_{\mathcal{F},M}$, i.e., the quotient of $\mathcal{T}_{M,S}$ by the image $i := \iota_S \circ \iota(\mathcal{F})$ in the following diagram

$$\begin{tikzcd} & & \mathcal{T}_S \arrow[d, "\iota_S"] & & \\ 0 \arrow[r] & \mathcal{F} \arrow[ur, "\iota"] \arrow[r, "i"] & \mathcal{T}_{M,S} \arrow[r, "\mathrm{pr}"] & \mathcal{N}_{\mathcal{F},M} \arrow[r] & 0. \end{tikzcd} \tag{3.4}$$

As we proved in last section the Atiyah class of $\mathcal{N}_{\mathcal{F},M}$, is the cohomological obstruction to the existence of a holomorphic connection on $N_{\mathcal{F},M}$ and is the obstruction to the splitting of the sequence:

$$0 \to \mathrm{Hom}(N_{\mathcal{F},M}, N_{\mathcal{F},M}) \to A_{N_{\mathcal{F},M}} \to TS \to 0, \tag{3.5}$$

where $A_{N_{\mathcal{F},M}}$ is the Atiyah sheaf of $N_{\mathcal{F},M}$. To compute the Atiyah class we have first to define the Atiyah sheaf for $N_{\mathcal{F},M}$ and compute its transition functions.

We first take the principal bundle P associated to $N_{\mathcal{F},M}$ (the bundle of $(n-l)$-frames of $N_{\mathcal{F},M}$). Suppose we are working on an atlas adapted to S and F (Definition 1.8.16); if $\{U_\alpha\}$ is a cover of S by coordinate neighborhoods in M (which are trivializing neighborhoods of $TM|_S$), these are going to be trivializing neighborhoods for $N_{\mathcal{F},M}$. Indeed, suppose

$$\left.\frac{\partial}{\partial z^1_\alpha}\right|_S, \dots, \left.\frac{\partial}{\partial z^n_\alpha}\right|_S$$

is a local frame for $TM|_S$, then we will denote by $\{\partial_{t,\alpha}\}$ the frame of $N_{\mathcal{F},M}$ induced by

$$\left.\frac{\partial}{\partial z^1_\alpha}\right|_S, \dots, \left.\frac{\partial}{\partial z^m_\alpha}\right|_S, \left.\frac{\partial}{\partial z^{m+l+1}_\alpha}\right|_S, \dots, \left.\frac{\partial}{\partial z^n_\alpha}\right|_S$$

and by $\{\omega^t_\alpha\}$ its dual frame. The transition functions for $N_{\mathcal{F},M}$ then have the form:

$$\partial_{t,\alpha} = \left.\frac{\partial z^w_\beta}{\partial z^t_\alpha}\right|_S \partial_{w,\alpha}.$$

We compute now the transition functions of P with respect to the cover $\{U_\alpha\}$: we have that a point $(x, (A)^t_w)$ in $U_\alpha \times \mathrm{GL}(n-l)$ is mapped to the frame

$$\{e_w = A^t_w \partial_{t,\alpha}\}$$

of $N_{\mathcal{F},M}|_x$. Changing trivialization neighborhood we have

$$e_w = A^t_w \left.\frac{\partial z^w_\beta}{\partial z^t_\alpha}\right|_S \partial_{w,\beta},$$

so that the point in $U_\beta \times \mathrm{GL}(n-l)$ is just

$$\left(x, \left(\left.\frac{\partial z^t_\beta}{\partial z^{w'}_\alpha}\right|_S \cdot A^{w'}_w\right)\right).$$

We have that $\mathrm{GL}(n-l)$ acts on the fibers of P preserving them by left action, which means on a trivializing neighborhood that given an element B of $\mathrm{GL}(n-l)$ and a point (x, A) in $U_\alpha \times \mathrm{GL}(n-l)$

$$L_B : (x, (A)^t_w) \mapsto (x, (B \cdot A)^t_w) = (x, B^t_{t'} \cdot A^{t'}_w).$$

The induced action of $\mathrm{GL}(n-l)$ on TP can be easily computed. On U_α we have a diffeomorphism

$$T(U_\alpha \times \mathrm{GL}(n-l)) = TU_\alpha \oplus TG = TU_\alpha \oplus \mathrm{GL}(n-l) \times \mathfrak{gl}(n-l) \simeq TP|_{U_\alpha}.$$

The induced left action of $\mathrm{GL}(n-l)$ on $TP|_{U_\alpha}$ can be computed in coordinates. Indeed, once we have chosen a trivialization of $N_{\mathcal{F},M}$, we can choose an atlas for $\mathrm{GL}(n-l)$ made by a single chart, whose coordinate functions are nothing else but:

$$A^t_w : A \mapsto (A)^t_w.$$

Let

$$\left\{\frac{\partial}{\partial A^t_w}\right\}_{t,w}$$

be the coordinate fields. Then, the left action of B on P induces:

$$(L_B)_*\left(G^t_w\frac{\partial}{\partial A^t_w}\right)=B^{t'}_tG^t_w\frac{\partial}{\partial A^{t'}_w}=(B\cdot G)^{t'}_w\frac{\partial}{\partial A^{t'}_w},$$

so the left action on TP is left multiplication on elements of the Lie algebra.

We compute now the transition functions for TP. We let U_α and U_β be trivializing neighborhoods as usual. Suppose on U_α we have a coordinate system (z_α, A_α) of $P|_{U_\alpha}$ and on U_β we have a coordinate system (z_β, A_β). From the computation of the transition functions of P we know that:

$$\begin{aligned} z^k_\beta &= z^k_\beta(z^1_\alpha,\dots,z^n_\alpha),\\ A^t_{w,\beta} &= \frac{\partial z^t_\beta}{\partial z^{w'}_\alpha}(z^1_\alpha,\dots,z^n_\alpha)\bigg|_S A^{w'}_{w,\alpha}. \end{aligned} \tag{3.6}$$

We compute now the transition matrix for TP. Because of the local triviality condition for principal bundles, it is going to be a block matrix. Indeed, from (3.6) we get:

$$\begin{aligned} \frac{\partial}{\partial A^t_{w,\alpha}} &= \frac{\partial z^{t'}_\beta}{\partial z^t_\alpha}\bigg|_S\frac{\partial}{\partial A^{t'}_{w,\beta}}\\ \frac{\partial}{\partial z^h_\alpha} &= \frac{\partial z^k_\beta}{\partial z^h_\alpha}\bigg|_S\frac{\partial}{\partial z^k_\beta}+\frac{\partial^2 z^t_\beta}{\partial z^h_\alpha\partial z^{w'}_\alpha}\bigg|_S A^{w'}_w\frac{\partial}{\partial A^t_{w,\beta}} \end{aligned}$$

And the transition matrix from $TP|_{U_\alpha}$ to $TP|_{U_\beta}$ has the form:

$$\begin{bmatrix} E & 0\\ F & G\end{bmatrix}$$

where:

$$E=\left(\frac{\partial z^h_\beta}{\partial z^k_\alpha}\bigg|_S\right),\quad F=\left(\frac{\partial^2 z^t_\beta}{\partial z^h_\alpha\partial z^{w'}_\alpha}\bigg|_S A^{w'}_w\right),\quad G=\left(\frac{\partial z^{t'}_\beta}{\partial z^t_\alpha}\bigg|_S\right).$$

We can finally compute the transition matrices for the Atiyah sheaf of $N_{\mathcal{F},M}$, defined as TP/G where the action we are referring to is the left action of $\mathrm{GL}(n-l)$ on P. Since we are quotienting by $\mathrm{GL}(n-l)$ we have that an equivalence class q can be represented in a trivialization by

$$(x,\xi^1,\dots,\xi^n,A,B)$$

as well as by

$$(x,\xi^1,\dots,\xi^n,\mathrm{id},A^{-1}B).$$

We have then that the transition matrix block F becomes:

$$F=\left(\frac{\partial^2 z^t_\beta}{\partial z^h_\alpha\partial z^w_\alpha}\bigg|_S\right).$$

Calling $C:=A^{-1}B$ and changing trivialization:

$$\left(x,\left(\frac{\partial z^k_\beta}{\partial z^h_\alpha}\bigg|_S\xi^h_\alpha\right),\left(\frac{\partial z^t_\beta}{\partial z^{t'}_\alpha}\bigg|_S\right),\left(\xi^k_\alpha\frac{\partial^2 z^t_\beta}{\partial z^k_\alpha\partial z^w_\alpha}\bigg|_S+C^{t'}_w\frac{\partial z^t_\beta}{\partial z^{t'}_\alpha}\bigg|_S\right)\right).$$

Again, passing to the quotient we can reduce the third entry to the identity:

$$\left(x, \frac{\partial z^k_\beta}{\partial z^i_\alpha}\bigg|_S \xi^i_\alpha, \mathrm{id}, \left(\xi^k_\alpha \frac{\partial z^{t'}_\alpha}{\partial z^t_\beta}\bigg|_S \frac{\partial^2 z^t_\beta}{\partial z^k_\alpha \partial z^w_\alpha}\bigg|_S + C^{t'}_w \frac{\partial z^{t'}_\alpha}{\partial z^t_\beta}\bigg|_S \frac{\partial z^t_\beta}{\partial z^{t'}_\alpha}\bigg|_S\right)\right) \tag{3.7}$$

$$= \left(x, \frac{\partial z^k_\beta}{\partial z^i_\alpha}\bigg|_S \xi^i_\alpha, \mathrm{id}, \left(\xi^k_\alpha \frac{\partial z^{t'}_\alpha}{\partial z^t_\beta}\bigg|_S \frac{\partial^2 z^t_\beta}{\partial z^k_\alpha \partial z^w_\alpha}\bigg|_S\right) + C^t_w\right) \tag{3.8}$$

Finally, we can compute the Atiyah class: the Atiyah class is the obstruction to find a splitting of the following short exact sequence:

$$0 \longrightarrow \mathrm{Hom}(N_{\mathcal{F},M}, N_{\mathcal{F},M}) \longrightarrow A_{N_{\mathcal{F},M}} \xrightarrow{\tilde{\pi}} TS \longrightarrow 0 \,,$$

where $\tilde{\pi}$ is the map induced by the bundle projection $\pi : P \to S$. This means that we have to compute the image in

$$H^1(\mathcal{U}, \mathrm{Hom}(TS, \mathrm{Hom}(N_{\mathcal{F},M}, N_{\mathcal{F},M})))$$

by the coboundary map δ^* (Theorem 1.3.9) of the identity in $H^0(\mathrm{Hom}(TM|_S, TM|_S))$:

$$\begin{aligned}
\delta^*\left\{U_\alpha, dz^k_\alpha \otimes \frac{\partial}{\partial z^k_\alpha}\right\} &= \left\{U_{\alpha\beta}, dz^k_\beta \otimes \frac{\partial}{\partial z^k_\beta} - dz^k_\alpha \otimes \frac{\partial}{\partial z^k_\alpha}\right\} \\
&= \Bigg\{U_{\alpha\beta}, dz^p_\beta \otimes \frac{\partial}{\partial z^p_\beta} - dz^q_\alpha \otimes \frac{\partial z^p_\beta}{\partial z^q_\alpha} \frac{\partial}{\partial z^p_\beta} \\
&\quad - \frac{\partial z^{t'}_\alpha}{\partial z^t_\beta} \frac{\partial^2 z^t_\beta}{\partial z^p_\alpha \partial z^w_\alpha}\bigg|_S dz^p_\alpha \otimes \omega^w_\beta \otimes \partial_{t',\beta}\Bigg\} \\
&= \left\{U_{\alpha\beta}, -\frac{\partial z^{t'}_\alpha}{\partial z^t_\beta} \frac{\partial^2 z^t_\beta}{\partial z^p_\alpha \partial z^w_\alpha}\bigg|_S dz^p_\alpha \otimes \omega^w_\beta \otimes \partial_{t',\beta}\right\}.
\end{aligned} \tag{3.9}$$

Following [4] and Section 3.1 we can define the partial Atiyah class. This class is defined as the obstruction to the splitting of the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(N_{\mathcal{F},M}, N_{\mathcal{F},M}) \longrightarrow A_{N_{\mathcal{F},M},F} \xrightarrow{\pi} F \longrightarrow 0 \,,$$

where $A_{N_{\mathcal{F},M},F} := \pi^{-1}(F)$. This cohomological class is the obstruction to the existence of a partial holomorphic connection with respect to F, which is the class:

$$\left\{U_{\alpha\beta}, -\frac{\partial z^{t'}_\alpha}{\partial z^t_\beta} \frac{\partial^2 z^t_\beta}{\partial z^i_\alpha \partial z^w_\alpha}\bigg|_S dz^i_\alpha \otimes \omega^w_\beta \otimes \partial_{t',\beta}\right\}, \tag{3.10}$$

where the only difference lies in the range of the indices, $p = m+1, \dots, n$ in (3.9) and $i = m+1, \dots, m+l$ in (3.10).

3.3 Tangent sheaves of infinitesimal neighborhoods

In this section we shall define what we mean by tangent sheaves of infinitesimal neighborhoods and study some of their properties. We are going to use the notion of logarithmic vectors field, introduced in [29]. The sheaf of these vector fields can behave badly if the subvariety S they leave invariant is non regular. In the rest of the section S is assumed to be a submanifold. We refer to Section 1.10 for the definition of k-th infinitesimal neighborhood and the notation.

Definition 3.3.1. A section v of $\mathcal{T}_M$ is called **logarithmic** if $v(\mathcal{I}_S) \subseteq \mathcal{I}_S$. The sheaf $\mathcal{T}_M(\log S) := \{v \in \mathcal{T}_M \mid v(\mathcal{I}_S) \subseteq \mathcal{I}_S\}$ is called the **sheaf of logarithmic sections** and is a subsheaf of $\mathcal{T}_M$. The **tangent sheaf of the k-th infinitesimal neighborhood**, denoted by $\mathcal{T}_{S(k)}$, is the image of the sheaf homomorphism $\mathcal{T}_M(\log S) \otimes_{\mathcal{O}_M} \mathcal{O}_{S(k)} \to \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{S(k)}$ and is a sheaf on S.

Remark 3.3.2. If a point x does not belong to S, the stalk $\mathcal{T}_M(\log S)_x$ coincides with $\mathcal{T}_{M,x}$. Suppose we have an atlas adapted to S; if $x \in S$ the stalk $\mathcal{T}_M(\log S)_x$ is generated by

$$z^r \frac{\partial}{\partial z^s}, \frac{\partial}{\partial z^p}.$$

Then a section v of $\mathcal{T}_{S(k)}$ is written locally as:

$$v = [a^r]_{k+1} \frac{\partial}{\partial z^r} + [a^p]_{k+1} \frac{\partial}{\partial z^p},$$

where the a^r belong to $\mathcal{I}_S$.

Remark 3.3.3. In the following, given a section v of $\mathcal{T}_{S(k)}$ and an open set U_α of M intersecting S, we shall denote by $\tilde{v}_\alpha$ a local extension of v to U_α as a section of $\mathcal{T}_M(U_\alpha)$; given an atlas adapted to S it is possible to build such an extension on each coordinate chart. If the open set is clear from the discussion we shall denote the extension simply by $\tilde{v}$. Please note that such an extension is not only a section of $\mathcal{T}_M(U_\alpha)$ but also a section of $\mathcal{T}_M(\log S)(U_\alpha)$. Taken an extension $\tilde{v}$, denoted by $[1]_{k+1}$ the class of 1 in $\mathcal{O}_{S(k)}(U_\alpha)$ we shall denote its restriction to the k-th infinitesimal neighborhood by

$$\tilde{v} \otimes [1]_{k+1}.$$

We prove in Lemma 3.3.4 that this notation is consistent with the fact that the sections of $\mathcal{T}_{S(k)}$ act as derivations of $\mathcal{O}_{S(k)}$. Moreover given two open sets U_α and U_β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$ and taken two extensions $\tilde{v}_\alpha$ and $\tilde{v}_\beta$ of a section v of $\mathcal{T}_{S(k)}$, respectively on U_α and U_β, it follows from the definition that on $U_\alpha \cap U_\beta$ we have the following equivalence:

$$v = \tilde{v}_\alpha \otimes [1]_{k+1} = \tilde{v}_\beta \otimes [1]_{k+1}. \tag{3.11}$$

Lemma 3.3.4. *The sections of $\mathcal{T}_{S(k)}$ act as derivations of $\mathcal{O}_{S(k)}$. Furthermore, given two sections v, w of $\mathcal{T}_{S(k)}$, their bracket, defined on each coordinate patch U_α such that $U_\alpha \cap S \neq \emptyset$ as*

$$[v, w] := [\tilde{v}_\alpha, \tilde{w}_\alpha] \otimes [1]_{k+1}$$

where the bracket on the right side is the usual bracket on $\mathcal{T}_M$, is a well defined section of $\mathcal{T}_{S(k)}$.

Proof. Let v be a section of $\mathcal{T}_{S(k)}$ and f a section of $\mathcal{O}_{S(k)}$. Let U_α and U_β two coordinate patches of an atlas adapted to S such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. On U_α we take representatives $\tilde{f}_1$ and $\tilde{f}_2$ of f and an extension $\tilde{v}_\alpha$ of v. We define:

$$v(f) = \tilde{v}_\alpha(\tilde{f}_1) \otimes [1]_{k+1} = [\tilde{v}_\alpha(\tilde{f}_1)]_{k+1}.$$

We check now that this does not depend on the extension chosen for f:

$$\tilde{v}_\alpha(\tilde{f}_1) - \tilde{v}_\alpha(\tilde{f}_2) = \tilde{v}_\alpha(\tilde{f}_1 - \tilde{f}_2) = \tilde{v}_\alpha(h_{r_1,\dots,r_{k+1}} z^{r_1} \dots z^{r_{k+1}}).$$

Since $\tilde{v}_\alpha$ is logarithmic, then

$$\tilde{v}_\alpha(h_{r_1,\dots,r_{k+1}} z^{r_1} \dots z^{r_{k+1}}) \in {\mathcal{I}_S}^{k+1}$$

and

$$(\tilde{v}_\alpha(\tilde{f}_1) - \tilde{v}_\alpha(\tilde{f}_2)) \otimes [1]_{k+1} = [0]_{k+1}.$$

Now, let $\tilde{v}_\alpha$ and $\tilde{v}'_\alpha$ be two extensions of v. Suppose $w_{1,\alpha}, \dots, w_{n,\alpha}$ are generators for $\mathcal{T}_M(U_\alpha)$. By definition:

$$\tilde{v}_\alpha - \tilde{v}'_\alpha = g^h_\alpha w_{h,\alpha},$$

with $g^h_\alpha \in {\mathcal{I}_S}^{k+1}$ for each h. Then:

$$(\tilde{v}_\alpha - \tilde{v}'_\alpha)(\tilde{f}_1) \otimes [1]_{k+1} = g^k_\alpha w_{k,\alpha}(\tilde{f}_1) \otimes [1]_{k+1} = w_{k,\alpha}(\tilde{f}_1) \otimes [g^k_\alpha]_{k+1} = [0]_{k+1}.$$

This implies also that if we take extensions $\tilde{v}_\alpha$ and $\tilde{v}_\beta$ and representatives $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ for f on U_α and U_β respectively we have that on $U_\alpha \cap U_\beta$ the derivation is well defined.

We prove now the bracket is well defined; if u and v are sections of $\mathcal{T}_{S(k)}$ the bracket is:

$$[u, v] = [\tilde{u}, \tilde{v}] \otimes [1]_{k+1}.$$

If $\tilde{u}_1, \tilde{u}_2$ are two extensions of u and $\tilde{v}_1, \tilde{v}_2$ are two extension of v then

$$\begin{aligned}[\tilde{u}_1, \tilde{v}_1] - [\tilde{u}_2, \tilde{v}_2] &= [\tilde{u}_1, \tilde{v}_1] - [\tilde{u}_1, \tilde{v}_2] + [\tilde{u}_1, \tilde{v}_2] - [\tilde{u}_2, \tilde{v}_2] \\ &= [\tilde{u}_1, \tilde{v}_1 - \tilde{v}_2] + [\tilde{u}_1 - \tilde{u}_2, \tilde{v}_2].\end{aligned}$$

As above, we have that

$$\tilde{u}_1 - \tilde{u}_2 = g^h_\alpha w_{h,\alpha}, \quad \tilde{v}_1 - \tilde{v}_2 = t^h_\alpha w_{h,\alpha},$$

with $g^h_\alpha, t^h_\alpha \in {\mathcal{I}_S}^{k+1}$ for every h. Then:

$$\begin{aligned}[\tilde{u}_1, \tilde{v}_1 - \tilde{v}_2] + [\tilde{u}_1 - \tilde{u}_2, \tilde{v}_2] &= [\tilde{u}_1, t^h_\alpha w_{h,\alpha}] + [g^h_\alpha w_{h,\alpha}, \tilde{v}_2] \\ &= \tilde{u}_1(t^h_\alpha) w_{h,\alpha} + t^h_\alpha [\tilde{u}_1, w_{h,\alpha}] - \tilde{v}_2(g^h_\alpha) w_{h,\alpha} + g^h_\alpha [w_{h,\alpha}, \tilde{v}_2]. \quad (3.12)\end{aligned}$$

Since both $\tilde{v}_1$ and $\tilde{u}_2$ are logarithmic, the restriction to the k-th infinitesimal neighborhood of (3.12) is 0. □

3.4 The concrete Atiyah sheaf for the normal bundle of a foliation in the ambient tangent bundle

The Atiyah sheaf is an important geometrical object first defined in [6]; we explicited the construction in Section 3.1. In Proposition 3.1.4 it is proved that the existence of a holomorphic connection for a vector bundle E is equivalent to the splitting of the following sequence:

$$0 \to \mathrm{Hom}(E,E) \to \mathcal{A}_E \to TM \to 0, \tag{3.13}$$

where $\mathcal{A}_E$ is the Atiyah sheaf of E; moreover we proved that the obstruction to the existence of partial holomorphic connections for a vector bundle E along a subbundle $\mathcal{F}$ is equivalent to the splitting of the following sequence:

$$0 \longrightarrow \mathrm{Hom}(E,E) \longrightarrow \mathcal{A}_{E,\mathcal{F}} \longrightarrow \mathcal{F} \longrightarrow 0\ . \tag{3.14}$$

Remark 3.4.1. In our case, we have to replace E with $N_{F,M}$; in Section 3.2 we computed this obstruction. In an atlas adapted to S and $\mathcal{F}$, in Čech-de Rham cohomology the class is represented by the cocycle

$$\left\{ U_{\alpha\beta}, -\frac{\partial z_\alpha^{t'}}{\partial z_\beta^t}\frac{\partial^2 z_\beta^t}{\partial z_\alpha^i \partial z_\alpha^w}\bigg|_S dz_\alpha^i \otimes \omega_\beta^w \otimes \partial_{t',\beta}\right\},$$

where $\{\partial_{t,\alpha}\}$ is the quotient frame for $N_{F,M}$ in U_α and ω_α^t is the dual frame for $N_{F,M}$ on U_α.

In paper [4] a more concrete version of the Atiyah sheaf is built; we follow their construction. We refer to Section 1.10 for the notation.

Definition 3.4.2. Let S be a not necessarily closed complex submanifold of M and $\mathcal{F}$ a foliation of S. Let $\mathcal{T}_{M,S(1)} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{S(1)}$ and $\mathcal{T}_{M,S} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$; if $\theta_1 : \mathcal{O}_{S(1)} \to \mathcal{O}_S$ is the canonical projection, we denote by Θ_1 the map $\mathrm{id} \otimes \theta_1 : \mathcal{T}_{M,S(1)} \to \mathcal{T}_{M,S}$. Let $\mathcal{T}^{\mathcal{F}}_{M,S(1)} := \ker(\mathrm{pr} \circ \Theta_1)$ **the formal extension of the foliation**, where pr is the quotient map in the short exact sequence:

$$\begin{tikzcd} & \mathcal{T}^{\mathcal{F}}_{M,S(1)} \arrow[r] & \mathcal{T}_{M,S(1)} \arrow[d, "\Theta_1"] & & \\ 0 \arrow[r] & \mathcal{F} \arrow[r, "i"] & \mathcal{T}_{M,S} \arrow[r, "\mathrm{pr}"] & \mathcal{N}_{\mathcal{F},M} \arrow[r] & 0. \end{tikzcd} \tag{3.15}$$

As in [4], we define a more concrete realization of the Atiyah sheaf for the sheaf $\mathcal{N}_{\mathcal{F},M}$.

Remark 3.4.3. By definition $\Theta_1(\mathcal{T}^{\mathcal{F}}_{M,S(1)})$ is contained in the kernel of pr, so, by exactness of sequence (3.4), it is contained in the image of $\mathcal{F}$. Moreover, for each $v \in \mathcal{F}$, at least locally, the element $\tilde{v} \otimes [1]_2$ belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ and is projected by Θ_1 to $i(v)$. So, $\Theta_1(\mathcal{T}^{\mathcal{F}}_{M,S(1)}) = i(\mathcal{F})$.

Remark 3.4.4. Suppose we have a coordinate system adapted to S and $\mathcal{F}$ (Definition 1.8.16). Then v belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ if and only if $v = [a^k]_2 \partial/\partial z^k$, with $[a^t]_1 = 0$, where $t = 1, \ldots, m, m+l+1, \ldots, n$. Analogously v belongs to $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ if and only if $v = [a^i]_2 \partial/\partial z^i$, where $a_1^i \in \mathcal{I}_S$ for $i = m+1, \ldots, m+l$.

Lemma 3.4.5. *Let S be a complex submanifold of a complex manifold M and $\mathcal{F}$ a foliation of S. Then*

1. *every v in $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ induces a derivation $g \mapsto v(g)$ of $\mathcal{O}_{S(1)}$;*

2. *there exist a natural $\mathbb{C}$-linear map* $\{\ \cdot\ ,\ \cdot\ \} : \mathcal{T}^{\mathcal{F}}_{M,S(1)} \otimes \mathcal{T}^{\mathcal{F}}_{M,S(1)} \to \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ *such that*

   (a) $\{u, v\} = -\{v, u\}$,

   (b) $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$,

   (c) $\{gu, v\} = g\{u, v\} - v(g)u,$, *for all* $g \in \mathcal{O}_{S(1)}$

   (d) $\Theta_1(\{u, v\}) = [\Theta_1(u), \Theta_1(v)]$.

Proof. 1. Let $(U; z^1, \dots, z^n)$ be a coordinate chart adapted to S and $\mathcal{F}$. An element $v = [a^k]_2 \frac{\partial}{\partial z^k} \in \mathcal{T}_{M,S(1)}$ belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ if and only if $[a^t]_1 = 0$. Remembering Remark 3.3.2 we see that v belongs to $\mathcal{T}_{S(1)}$ and Lemma 3.3.4 gives the assertion.

2. We define $\{\ \cdot\ ,\ \cdot\ \}$ by setting

$$\{u, v\}(f) = u(v(f)) - v(u(f)),$$

for every $f \in \mathcal{O}_{S(1)}$. Please note that, since u and v belong to $\mathcal{T}_{S(1)}$ this bracket coincides with the bracket defined on $\mathcal{T}_{S(1)}$; the first three properties are proved exactly as for the usual bracket of vector fields, while the fourth follows from a simple computation in coordinates. Suppose $(U; z^1, \dots, z^n)$ is a coordinate chart adapted to S and $\mathcal{F}$, $u = [a^k]_2 \frac{\partial}{\partial z^k}$, $v = [b^k]_2 \frac{\partial}{\partial z^k}$ with $[a^t]_1 = 0$ and $[b^t]_1 = 0$. First of all we compute the Lie brackets on $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ in coordinates:

$$\begin{aligned}
\{u, v\} =& \left[a^h \frac{\partial b^k}{\partial z^h} - b^h \frac{\partial a^k}{\partial z^h} \right]_2 \frac{\partial}{\partial z^k} \\
=& \left[a^t \frac{\partial b^u}{\partial z^t} + a^i \frac{\partial b^u}{\partial z^i} - b^t \frac{\partial a^u}{\partial z^t} - b^i \frac{\partial a^u}{\partial z^i} \right]_2 \frac{\partial}{\partial z^u} \\
&+ \left[a^t \frac{\partial b^j}{\partial z^t} - b^t \frac{\partial a^j}{\partial z^t} \right]_2 \frac{\partial}{\partial z^j} + \left[a^i \frac{\partial b^j}{\partial z^i} - b^i \frac{\partial a^j}{\partial z^i} \right]_2 \frac{\partial}{\partial z^j}.
\end{aligned}$$

Please note that the coefficients in the first two summands of the last expression all belong to $\mathcal{I}_S/\mathcal{I}_S^2$. Therefore:

$$\Theta_1(\{u, v\}) = \left[a^i \frac{\partial b^j}{\partial z^i} - b^i \frac{\partial a^j}{\partial z^i} \right]_1 \frac{\partial}{\partial z^j} = [\Theta_1(u), \Theta_1(v)].$$

□

Remark 3.4.6. In general, given two vector fields u, v in $\mathcal{T}_{M,S(1)}$, we can define a bracket as $[u, v](f) = u(v(f)) - v(u(f))$, for $f \in \mathcal{O}_{S(1)}$. Please note that this bracket is not a well defined section of $\mathcal{T}_{M,S(1)}$ but only of $\mathcal{T}_{M,S}$. In other words $[u(v(f)) - v(u(f))]_2$ is not well defined, while $[u(v(f)) - v(u(f))]_1$ is.

Lemma 3.4.7. *Let S be an m-codimensional complex submanifold of a complex manifold M of complex dimension n and $\mathcal{F}$ a foliation of S. Then:*

1. $u \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ *is such that* $pr([u, s]) = 0$ *for all* $s \in \mathcal{T}_{M,S(1)}$ *if and only if* $u \in \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$;

2. if $u \in \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ and $v \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ then $\{u,v\} \in \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$;

3. the quotient sheaf

$$\mathcal{A} = \mathcal{T}^{\mathcal{F}}_{M,S(1)} / \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$$

admits a natural structure of $\mathcal{O}_S$-locally free sheaf such that the map induced by Θ_1, whose image lies in $\mathcal{F}$, is an $\mathcal{O}_S$-morphism.

Proof. 1. Writing $u = [a^k]_2 \frac{\partial}{\partial z^k}$, with $[a^t]_1 = 0$, and $s = [b^h]_2 \frac{\partial}{\partial z^h} \in \mathcal{T}_{M,S(1)}$, we have:

$$\mathrm{pr}([u,s]) = \left[a^k \frac{\partial b^t}{\partial z^k} - b^k \frac{\partial a^t}{\partial z^k}\right]_1 \frac{\partial}{\partial z^t}.$$

If u belongs to $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ clearly $\mathrm{pr}([u,s]) = 0$.
Conversely, let u be such that $\mathrm{pr}([u,s]) = 0$ for each $s \in \mathcal{T}_{M,S(1)}$. We claim it belongs to $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$. We know that u belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$, so $[a^t]_1 = 0$. Taking $s = \partial/\partial z^r$, we see that $[\partial a^t/\partial z^r]_1 = 0$. Now, we take a representative $h_s z^s$ for the class $[a^t]_1$. Computing:

$$0 = \left[\frac{\partial a^t}{\partial z^r}\right]_1 = \left[\frac{\partial h_s}{\partial z^s} z^s + h_s \delta^s_r\right]_1 = [h_s]_1.$$

So, for each s, we have that h_s belongs to $\mathcal{I}_S$, implying that $[a^t]_2 = 0$. Let now $s = [z^j]_2 \frac{\partial}{\partial z^1}$, for $j = m+1, \ldots, n$. Then

$$0 = -\left[z^j \frac{\partial a^t}{\partial z^1}\right]_1 \frac{\partial}{\partial z^t} + \left[a^k \delta^j_k\right]_1 \frac{\partial}{\partial z^1} = [a^j]_1 \frac{\partial}{\partial z^1},$$

where the last equality follows from the preceeding step, where we proved that $[a^t]_2 = 0$ and thus that $\left[\frac{\partial a^t}{\partial z^1}\right]_1 = 0$. So, $[a^j]_1 = 0$ and u belongs to $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$.

2. This follows by a direct computation in coordinates.

3. The sheaf $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ is an $\mathcal{O}_{S(1)}$-submodule of $\mathcal{T}_{M,S(1)}$ such that $g \cdot v$ belongs to $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ for every $g \in \mathcal{I}_S/\mathcal{I}_S^2$ and $v \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$. Therefore the $\mathcal{O}_{S(1)}$ structure induces a natural $\mathcal{O}_S$-module structure on $\mathcal{A}$.
Recall $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ is generated locally, in an atlas adapted to S, by $\partial/\partial z^j$, with $j = m+1, \ldots, m+l$ and by $[z^r]_2 \partial/\partial z^s$, with r and s varying in $1, \ldots, m$. Then, the sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_S$-module freely generated by $\pi(\frac{\partial}{\partial z^j})$ and $\pi([z^s]_2 \frac{\partial}{\partial z^t})$, where $\pi : \mathcal{T}^{\mathcal{F}}_{M,S(1)} \to \mathcal{A}$ is the quotient map. Moreover, $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ lies in the kernel of Θ_1 so Θ_1 factors through a map that we will denote again by $\Theta_1 : \mathcal{A} \to \mathcal{F}$, which is clearly an $\mathcal{O}_S$-morphism.

□

Definition 3.4.8. Let S be a complex submanifold of a complex manifold M and let $\mathcal{F}$ be a foliation of S. The **Atiyah sheaf of** $\mathcal{F}$ is the locally free $\mathcal{O}_S$-module

$$\mathcal{A} = \mathcal{T}^{\mathcal{F}}_{M,S(1)} / \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}\,.$$

Theorem 3.4.9. *Let S be a codimension m submanifold of a complex manifold M of dimension n and let $\mathcal{F}$ be a foliation of S. Then there exists a natural exact sequence of locally free $\mathcal{O}_S$-modules*

$$0 \longrightarrow Hom(\mathcal{N}_S, \mathcal{N}_{\mathcal{F},M}) \longrightarrow \mathcal{A} \xrightarrow{\Theta_1} \mathcal{F} \longrightarrow 0\,.$$

whose splitting is equivalent to the splitting of the sequence (3.14) *taking $\mathcal{N}_{F,M}$ as E.*

Proof. We work in a chart adapted to S and $\mathcal{F}$. The kernel of Θ_1 is locally freely generated by the images under $\pi : \mathcal{T}^{\mathcal{F}}_{M,S(1)} \to \mathcal{A}$ of $[z^s_\alpha]_2 \frac{\partial}{\partial z^t_\alpha}$. We would like to understand how the generators behave under change of coordinates, to see if $\ker(\Theta_1)$ is isomorphic to any known sheaf of sections of a known vector bundle. We compute the coordinate change maps:

$$\begin{aligned}
\pi\left([z^s_\alpha]_2 \frac{\partial}{\partial z^t}\right) &= \pi\left([z^s_\alpha]_2 \left[\frac{\partial z^k_\beta}{\partial z^t_\alpha}\right]_2 \frac{\partial}{\partial z^k_\beta}\right) = \pi\left([z^s_\alpha]_2 \left[\frac{\partial z^w_\beta}{\partial z^t_\alpha}\right]_2 \frac{\partial}{\partial z^w_\beta}\right) &(3.16)\\
&= \pi\left(\left[\frac{\partial z^s_\alpha}{\partial z^{s_1}_\beta}\right]_2 [z^{s_1}_\beta]_2 \left[\frac{\partial z^w_\beta}{\partial z^t_\alpha}\right]_2 \frac{\partial}{\partial z^w_\beta}\right)\\
&= \left[\frac{\partial z^s_\alpha}{\partial z^{s_1}_\beta}\frac{\partial z^w_\beta}{\partial z^t_\alpha}\right]_1 \pi\left([z^{s_1}_\beta]_2 \frac{\partial}{\partial z^w_\beta}\right), &(3.17)
\end{aligned}$$

where the last equality in (3.16) comes from the quotient map and the one in (3.17) comes from the newly acquired structure of $\mathcal{O}_S$-module. As a consequence, the kernel of Θ_1 is isomorphic to $\mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_{\mathcal{F},M})$.

Now, if we define local splittings of the sequence by setting

$$\sigma_\alpha\left(\frac{\partial}{\partial z^j_\alpha}\right) = \pi\left(\frac{\partial}{\partial z^j_\alpha}\right)$$

and extending by $\mathcal{O}_S$-linearity, we can compute the obstruction to find a splitting of the sequence:

$$\begin{aligned}
(\sigma_\beta - \sigma_\alpha)\left(\frac{\partial}{\partial z^j_\beta}\right) &= \sigma_\beta\left(\frac{\partial}{\partial z^j_\beta}\right) - \sigma_\alpha\left(\left[\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right]_1 \frac{\partial}{\partial z^i_\alpha}\right)\\
&= \sigma_\beta\left(\frac{\partial}{\partial z^j_\beta}\right) - \left[\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right]_1 \sigma_\alpha\left(\frac{\partial}{\partial z^i_\alpha}\right)\\
&= \pi\left(\frac{\partial}{\partial z^j_\beta}\right) - \left[\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right]_1 \pi\left(\frac{\partial}{\partial z^i_\alpha}\right)\\
&= \pi\left(\left[\frac{\partial z^t_\alpha}{\partial z^j_\beta}\right]_2 \frac{\partial}{\partial z^t_\alpha}\right) = \pi\left(\left[\frac{\partial^2 z^t_\alpha}{\partial z^r_\beta \partial z^j_\beta} z^r_\beta\right]_2 \frac{\partial}{\partial z^t_\alpha}\right)\\
&= \left[\frac{\partial^2 z^t_\alpha}{\partial z^r_\beta \partial z^j_\beta}\frac{\partial z^r_\beta}{\partial z^s_\alpha}\right]_1 \pi\left([z^s_\alpha]_2 \frac{\partial}{\partial z^t_\alpha}\right). &(3.18)
\end{aligned}$$

Please remark that, since $\partial z^t_\alpha / \partial z^j_\beta$ lies in the ideal $\mathcal{I}_S$ for $t = 1, \ldots, m, m+l+1, \ldots, n$ and $j = 1, \ldots, n$ it follows that

$$\frac{\partial z^t_\alpha}{\partial z^p_\beta \partial z^j_\beta} \in \mathcal{I}_S$$

for $t = 1, \dots, m, m+l+1, \dots, n$, $j = 1, \dots, n$ and $p = m+1, \dots, n$. Therefore we have that

$$\left[\frac{\partial z^t_\alpha}{\partial z^w_\beta \partial z^j_\beta}\right]_1 = [0]_1$$

for $t, w = 1, \dots, m, m+l+1, \dots, n$ and $j = 1, \dots, n$ if and only if

$$\left[\frac{\partial z^t_\alpha}{\partial z^r_\beta \partial z^j_\beta}\right]_1 = [0]_1$$

for $t, w = 1, \dots, m, m+l+1, \dots, n$, $j = 1, \dots, n$ and $r = 1, \dots, m$. Hence, class (3.18) vanishes if and only if (3.10) vanishes. □

It is easily noted that in the case $\mathcal{F}$ is the tangent bundle to S the Atiyah sheaf of $\mathcal{F}$ is nothing else than the Atiyah sheaf of S, defined in [4].

Definition 3.4.10. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules over a complex manifold S, equipped with a $\mathcal{O}_S$-morphism $X : \mathcal{F} \to \mathcal{T}_S$. We say that $\mathcal{F}$ is a **Lie algebroid of anchor** X if there is a $\mathbb{C}$-bilinear map $\{\,\cdot\,,\,\cdot\,\} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{F}$ such that:

1. $\{v, u\} = -\{u, v\}$;
2. $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$;
3. $\{g \cdot u, v\} = g \cdot \{u, v\} - X(v)(g) \cdot u$ for all $g \in \mathcal{O}_S$ and $u, v \in \mathcal{F}$.

Definition 3.4.11. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of $\mathcal{O}_S$-modules over a complex manifold S. Given a section $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$, a **holomorphic** X**-connection on** $\mathcal{E}$ is a $\mathbb{C}$-linear map $\tilde{X} : \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that:

$$\tilde{X}(g \cdot s) = X^*(dg) \otimes s + g\tilde{X}(s),$$

for each $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$, where X^* is the dual map of X. The notation $\tilde{X}_v(s)$ is equivalent to $\tilde{X}(s)(v)$.

If $\mathcal{F}$ is a Lie algebroid of anchor X we define the **curvature** of $\tilde{X}$ to be:

$$R_{u,v}(s) = \tilde{X}_u \circ \tilde{X}_v(s) - \tilde{X}_v \circ \tilde{X}_u(s) - \tilde{X}_{\{u,v\}}(s).$$

We say that $\tilde{X}$ is **flat** if $R \equiv 0$.

Proposition 3.4.12. *Let S be a complex submanifold of a complex manifold M and $\mathcal{F}$ a holomorphic foliation of S. Then:*

1. *the Atiyah sheaf of $\mathcal{F}$ has a natural structure of Lie algebroid of anchor Θ_1 such that*
$$\Theta_1\{q_1, q_2\} = [\Theta_1(q_1), \Theta_1(q_2)]$$
for all $q_1, q_2 \in \mathcal{A}$;
2. *there is a natural holomorphic Θ_1-connection $\tilde{X} : \mathcal{N}_{\mathcal{F},M} \to \mathcal{A}^* \otimes \mathcal{N}_{\mathcal{F},M}$ on $\mathcal{N}_{\mathcal{F},M}$ given by*
$$\tilde{X}_q(s) = pr([v, \tilde{s}])$$
for all $q \in \mathcal{A}$ and $s \in \mathcal{N}_{\mathcal{F},M}$, where $v \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ and $\tilde{s} \in \mathcal{T}_{M,S(1)}$ are such that $\pi(v) = q$ and $pr \circ \Theta_1(\tilde{s}) = s$;

3. *this holomorphic Θ_1-connection is flat.*

Proof. 1. We set

$$\{q_1, q_2\} = \pi(\{v_1, v_2\}),$$

where $v_i \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ are such that $q_i = \pi(v_i)$, for $i = 1, 2$. This is well defined: if $q_1 = 0$, then v_1 is in $\mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$ and then, by 2 of Lemma 3.4.7 we have that $\{q_1, q_2\} = 0$. The other properties follow directly from Lemma 3.4.5.

2. We check the connection is well defined. Suppose now $q = 0$; this means that $v \in \mathcal{I}_S\, \mathcal{T}^{\mathcal{F}}_{M,S(1)}$; then, by Lemma 3.4.7.1, we have that $\mathrm{pr}([v, \tilde{s}]) = 0$, for every $\tilde{s} \in \mathcal{T}_{M,S(1)}$. Now, if $\mathrm{pr} \circ \Theta_1(\tilde{s}) = 0$, we have that $\tilde{s} \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$, so $\{v, \tilde{s}\}$ is in $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$, which implies that $\tilde{X}_q(s) = 0$.
We check now it is a Θ_1-connection. It is $\mathcal{O}_S$-linear in the first entry since:

$$\tilde{X}_{[f]_1 \cdot q}(s) = \mathrm{pr}([[f]_2 v, \tilde{s}]) = \mathrm{pr}([f]_1 [v, \tilde{s}] - \tilde{s}([f]_2)\Theta_1(v)) = [f]_1 \tilde{X}_q(v),$$

where the last equality comes from the fact that v belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$, which is the kernel of $\mathrm{pr} \circ \Theta_1$. We check the Θ_1-Leibniz rule for the second entry:

$$\begin{aligned}\tilde{X}_q([f]_1 s) &= \mathrm{pr}([v, [f]_2 \tilde{s}]) = \mathrm{pr}([f]_1 [v, \tilde{s}] + v([f]_2) \cdot \Theta_1(\tilde{s})) \\ &= [f]_1 \tilde{X}_q(s) + \Theta_1(q)([f]_1) \cdot s,\end{aligned}$$

where the last equality comes from the equality:

$$[v([f]_2)]_1 = \Theta_1(v)([f]_1) = \Theta_1(\pi(v))([f]_1),$$

for every $[f]_2 \in \mathcal{O}_{S(1)}$ and for every $v \in \mathcal{T}^{\mathcal{F}}_{M,S(1)}$. Thus, $\tilde{X}$ is a holomorphic Θ_1-connection.

3. We compute the curvature:

$$\begin{aligned}R_{q_1,q_2}(s) &= \tilde{X}_{q_1} \circ \tilde{X}_{q_2}(s) - \tilde{X}_{q_2} \circ \tilde{X}_{q_1}(s) - \tilde{X}_{\{q_1,q_2\}}(s) \\ &= \mathrm{pr}([u, \widetilde{\mathrm{pr}([v, \tilde{s}])}]) - \mathrm{pr}([v, \widetilde{\mathrm{pr}([u, \tilde{s}])}]) - \mathrm{pr}([[u, v], \tilde{s}]).\end{aligned}$$

As we proved before, the connection does not depend on the extension chosen for the second entry, so we can rewrite the expression as:

$$\mathrm{pr}([u, [v, \tilde{s}]]) - \mathrm{pr}([v, [u, \tilde{s}]]) - \mathrm{pr}([[u, v], \tilde{s}]).$$

Computing in coordinates, it follows from the usual Jacobi identity for vector fields that it is identically 0.

□

Definition 3.4.13. Let S be a complex submanifold of a complex manifold M and $\mathcal{F}$ a foliation of S. The holomorphic Θ_1- connection $\tilde{X} : \mathcal{N}_{\mathcal{F},M} \to \mathcal{A}^* \otimes \mathcal{N}_{\mathcal{F},M}$ just introduced is called the **universal holomorphic connection** on $\mathcal{N}_{\mathcal{F},M}$.

Remark 3.4.14. In the next chapter we find some conditions under which the universal holomorphic connection induces a partial holomorphic connection along a subbundle on $\mathcal{N}_{\mathcal{F},M}$.

Chapter 4

Holomorphic foliations

Remark 4.0.15. In this chapter we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

4.1 Foliations of infinitesimal neighborhoods

We have seen in Section 3.3 how to define the tangential sheaf to an infinitesimal neighborhood and how on this sheaf there exists a well defined bracket operation. We are going to use the notion of logarithmic vectors field, introduced in [29]. The sheaf of these vector fields can behave badly if the subvariety S they leave invariant is non regular. In the rest of the section S is assumed to be a submanifold. We refer to Section 1.10 for the definition of k-th infinitesimal neighborhood and the notation.

Therefore, the following definition makes sense.

Definition 4.1.1. Let M be a complex manifold of dimension n and S be a complex submanifold of codimension m. A **regular foliation** of $S(k)$ is a rank l (with $l \leq n-m$) coherent subsheaf $\mathcal{F}$ of $\mathcal{T}_{S(k)}$, such that:

- for every $x \in S$ the stalk $\mathcal{T}_{S(k)} \,/\, \mathcal{F}_x$ is $\mathcal{O}_{S(k),x}$-free;
- for every $x \in S$ we have that $[\mathcal{F}_x, \mathcal{F}_x] \subseteq \mathcal{F}_x$ (where the bracket is the one defined in Lemma 3.3.4);
- the restriction of $\mathcal{F}$ to S, denoted by $\mathcal{F}\,|_S$, is a rank l foliation of S.

Remark 4.1.2. Please note that the third condition is a simplifying condition: in the paper [12] a lot of work is devoted to clarify and explain the concept of extension of a foliation and our definition is a particular case. We want to avoid the following situation: let U be an open neighborhood of the origin in $\mathbb{C}^2$, with coordinate system (z_1, z_2) and let $S = z_1 = 0$. We take a subbundle of $\mathcal{T}_{S(1)}$ generated by $[z_1]_2\partial/\partial z_1, \partial/\partial z_2$. Clearly, it is involutive with respect to the bracket defined above, but its restriction to S gives rise to a rank 1 foliation.

The main tool of this section is the Holomorphic Frobenius Theorem 1.8.9. Lemma 4.1.3 is a tool we use in proving the Frobenius Theorem for foliations of the k-th infinitesimal neighborhood.

Lemma 4.1.3. *Every regular foliation $\mathcal{F}$ of $S(k)$ admits a local frame which can be extended locally by commuting vector fields, i.e., for every point $x \in S$ there exists a neighborhood U_x of x in M and commuting sections $\tilde{w}_{m+1}, \ldots, \tilde{w}_{m+l}$ of $\mathcal{T}_M$ on U_x such that $w_i := \tilde{w}_i \otimes [1]_{k+1}$ are generators of $\mathcal{F}(U_x \cap S)$.*

Proof. Let x be a point of S; we take a coordinate patch (U, ϕ) centered in x, adapted to S and $\mathcal{F}|_S$ (Definition 1.8.16). Let $\{v_i\}$ be a system of generators of $\mathcal{F}$ in U_x and $\{\tilde{v}_i\}$ be vector fields extending them. Call D the distribution spanned by the $\tilde{v}_i$'s. We complete the frame $\{\tilde{v}_i\}$ to a frame $\{\tilde{v}_k\}$ of TM, taking as $\tilde{v}_t$ the coordinate fields $\partial/\partial z^t$. Now, we choose holomorphic functions f_i^k such that:

$$\tilde{v}_k = f_k^h \frac{\partial}{\partial z^h}.$$

Please remark that the matrix $A := (f_k^h)$ is a matrix of holomorphic functions acting on the right:

$$|\tilde{v}^1, \ldots, \tilde{v}^n| = \left| \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right| \cdot A.$$

By hypothesis we know that A is non singular in x, so there exists a neighborhood U of x such that this matrix is invertible with inverse a matrix of holomorphic functions. Let (g_h^k) be its inverse matrix. We define $\tilde{w}_i = g_i^j \tilde{v}_j$ and we denote by $w_i := \tilde{w}_i \otimes [1]_{k+1}$. Each one of the $\tilde{w}_i$'s belongs to the module generated by $\tilde{v}_{m+1}, \ldots, \tilde{v}_{m+l}$, so leaves the ideal of S invariant. This implies, thanks to Lemma 3.3.4, that

$$[w_i, w_j] = [\tilde{w}_i, \tilde{w}_j] \otimes [1]_{k+1} = [g_i^{i'} \tilde{v}_{i'}, h_j^{j'} \tilde{v}_{j'}] \otimes [1]_{k+1} \in \mathcal{F}.$$

We claim now that the $\tilde{w}_j$ generate D and therefore, when restricted to $S(k)$ generate $\mathcal{F}$. Let π be the projection $(z^1, \ldots, z^n) \mapsto (z^{m+1}, \ldots, z^{m+l})$ and $\Pi = \pi \circ \phi$. We have:

$$\Pi_*(\tilde{w}_i) = \Pi_*(\tilde{w}_i) + g_i^t \Pi_* \left(\frac{\partial}{\partial z^t} \right) = \Pi_*(g_i^k \tilde{v}_k) = \Pi_* \left(\frac{\partial}{\partial z^i} \right) = \frac{\partial}{\partial z^i},$$

so the $\tilde{w}_i$ generate D. Moreover, by naturality of Lie brackets, we have that

$$\Pi_*([\tilde{w}_i, \tilde{w}_j]) = [\Pi_*(\tilde{w}_i), \Pi_*(\tilde{w}_j)].$$

The mapping Π_* induces a map $\Pi_{*,k} : \mathcal{T}_M \otimes \mathcal{O}_{S(k)} \to \mathcal{O}^l_{S(k)}$, given by:

$$\Pi_{*,k}(v \otimes [1]_{k+1}) = \Pi_*(\tilde{v}) \otimes [1]_{k+1}.$$

This map is injective when restricted to $\mathcal{F}$; since $[w_i, w_j] \in \mathcal{F}$ and $\Pi_{*,k}([w_i, w_j]) = 0$ we have that $[w_i, w_j] = 0$. We want now to modify the $\tilde{w}_i$'s to obtain l independent commuting sections of $\mathcal{F}$, without changing their equivalence class. Therefore, we look for extensions of the w_i's which satisfy the thesis of the theorem, proceeding by induction on the number of sections. If $l' = 1$, we can take any extension of w_{m+1} (every vector field commutes with itself). Suppose now the claim is true for $l' - 1$ sections. Then, by the Holomorphic Frobenius theorem, there exists a coordinate chart adapted to S in which

$\tilde{w}_{m+1} = \partial/\partial z^{m+1}, \ldots, \tilde{w}_{m+l'-1} = \partial/\partial z^{m+l'-1}$. Now, since the w_i are commuting when restricted to $S(k)$, if

$$w_{m+l'} = [g^v]_{k+1}\frac{\partial}{\partial z^v} + [f^i]_{k+1}\frac{\partial}{\partial z^i},$$

we have that:

$$[0]_{k+1} = \frac{\partial [g^v]_{k+1}}{\partial z^i}\frac{\partial}{\partial z^v} + \frac{\partial [f^j]_{k+1}}{\partial z^i}\frac{\partial}{\partial z^j} = \left[\frac{\partial g^v}{\partial z^i}\right]_{k+1}\frac{\partial}{\partial z^v} + \left[\frac{\partial f^j}{\partial z^i}\right]_{k+1}\frac{\partial}{\partial z^j},$$

where i ranges in $m+1, \ldots, m+l'-1$. The last equality tells us that:

$$\frac{\partial g^v}{\partial z^i} = z^{r_1}\cdots z^{r_{k+1}} h^v_{r_1,\cdots,r_{k+1},i}\,, \qquad \frac{\partial f^j}{\partial z^i} = z^{r_1}\cdots z^{r_{k+1}} h^j_{r_1,\cdots,r_{k+1},i}.$$

We have to find $\tilde{g^v}$, $\tilde{f^j}$ representatives for the classes $[g^v]_{k+1}$,$[f^j]_{k+1}$ such that

$$0 = \frac{\partial \tilde{g}^v}{\partial z^i}\frac{\partial}{\partial z^v} + \frac{\partial \tilde{f}^j}{\partial z^i}\frac{\partial}{\partial z^j}.$$

We do that for one of the g^v's, the method applies to all the other coefficients. Now, $\tilde{g}^v = g^v + z^{r_1}\cdots z^{r_{k+1}}\tilde{h}_{r_1,\ldots,r_{k+1}}$, so

$$\begin{aligned}\frac{\partial \tilde{g}^v}{\partial z^i} &= \frac{\partial g^v}{\partial z^i} + z^{r_1}\cdots z^{r_{k+1}}\frac{\partial \tilde{h}^v_{r_1,\ldots,r_{k+1}}}{\partial z^i}\\ &= z^{r_1}\cdots z^{r_{k+1}} h^v_{r_1,\ldots,r_{k+1},i} + z^{r_1}\cdots z^{r_{k+1}}\frac{\partial \tilde{h}^v_{r_1,\ldots,r_{k+1}}}{\partial z^i}.\end{aligned}$$

Therefore, the problem reduces to finding a primitive $\tilde{h}^v_{r_1,\ldots,r_{k+1}}$ for the 1-form

$$\omega := -h^v_{r_1,\ldots,r_{k+1},i}dz^i,$$

where the other coordinates are considered as parameters. If we denote by ∂ the holomorphic differential and supposing, without loss of generality, that U is simply connected and centered at $x \in S$ (i.e. $\phi(x) = 0$) we have, by the Holomorphic Poincaré Lemma, that this primitive exists if and only if ω is closed. Therefore we need to check that the mixed partial derivatives coincide:

$$z^{r_1}\cdots z^{r_{k+1}}\frac{\partial h^v_{r_1,\ldots,r_{k+1},i}}{\partial z^j} = \frac{\partial^2 g^v}{\partial z^j \partial z^i} = \frac{\partial^2 g^v}{\partial z^i \partial z^j} = z^{r_1}\cdots z^{r_{k+1}}\frac{\partial h^v_{r_1,\ldots,r_{k+1},j}}{\partial z^i}.$$

Then, the primitive exists and is defined in U by:

$$\tilde{h}^v_{r_1,\ldots,r_{k+1}}(z^1,\ldots,z^n) = \int_\gamma -h^v_{r_1,\ldots,r_{k+1},i}dz^i,$$

where γ is a curve such that $\gamma(1) = (z^1,\ldots,z^n)$ and $\gamma(0) = 0$. □

As a simple consequence of the Lemma, we have the Frobenius Theorem for k-th infinitesimal neighborhoods.

Theorem 4.1.4 (Frobenius Theorem for k-th infinitesimal neighborhoods)**.** *Suppose S is a non singular complex submanifold of codimension m in a complex manifold M of dimension n and suppose we have a regular foliation $\mathcal{F}$ of $S(k)$ of rank l. Then there exists an atlas $\{U_\alpha, \phi_\alpha\}$ adapted to S such that if $U_\alpha \cap U_\beta \cap S \neq \emptyset$ then:*

$$\left[\frac{\partial z_\alpha^t}{\partial z_\beta^i}\right]_{k+1} = 0, \tag{4.1}$$

for $t = 1, \dots, m, m+l+1, \dots, n$ and $i = m+1, \dots, m+l$ on $U_\alpha \cap U_\beta$.

Proof. We take an atlas adapted to S and extensions $\tilde{w}_{i,\alpha}$ as given by Lemma 4.1.3. By the usual Holomorphic Frobenius theorem, there exist a coordinate system (modulo shrinking) on U_α such that

$$\tilde{w}_{m+1,\alpha} = \frac{\partial}{\partial z_\alpha^{m+1}}, \dots, \tilde{w}_{m+l,\alpha} = \frac{\partial}{\partial z_\alpha^{m+l}}.$$

We take such coordinate systems. Since we are dealing with a foliation of $S(k)$ we have that $w_{i,\alpha} = [c_i^j]_{k+1} w_{j,\beta}$. Hence:

$$\begin{aligned}\left[\frac{\partial z_\alpha^t}{\partial z_\beta^i}\right]_{k+1} &= \tilde{w}_{i,\beta} \otimes [1]_{k+1}(z_\alpha^t) = w_{i,\beta}(z_\alpha^t) = [c_i^j]_{k+1} w_{j,\alpha}(z_\alpha^t) \\ &= [c_i^j]_{k+1} \tilde{w}_{j,\alpha} \otimes [1]_{k+1}(z_\alpha^t) = [c_i^j]_{k+1} \left[\frac{\partial z_\alpha^t}{\partial z_\alpha^j}\right]_{k+1} = [c_i^j \delta_j^t]_{k+1} = [0]_{k+1}.\end{aligned}$$

□

Remark 4.1.5. It is easily seen that the existence of an atlas satisfying (4.1) implies the existence of a foliation of $\mathcal{T}_{S(k)}$, generated on each chart U_α intersecting S by $\{[1]_{k+1} \otimes \partial/\partial z_\alpha^{m+1}, \dots, [1]_{k+1} \otimes \partial/\partial z_\alpha^{m+l}\}$.

Definition 4.1.6. We say that a foliation $\mathcal{F}$ of S **extends to the k-th infinitesimal neighborhood**, if there exists an atlas adapted to S and $\mathcal{F}$ such that:

$$\left[\frac{\partial z_\beta^t}{\partial z_\alpha^i}\right]_{k+1} = 0,$$

for $t = 1, \dots, m, m+l+1, \dots, n$ and $i = m+1, \dots, m+l$ on $U_\alpha \cap U_\beta$.
In the special case $\mathcal{F} = \mathcal{T}_S$ we say that S has **k-th order extendable tangent bundle**.

Remark 4.1.7. Let M be a complex manifold, $\mathcal{F}$ a regular foliation of M. Every leaf of $\mathcal{F}$ has k-th order extendable tangent bundle for every k.

Remark 4.1.8. For a submanifold S having first order extendable tangent bundle is a strong topological condition. As a matter of fact, as we will see in section 5.4, this implies the vanishing of many of the characteristic classes of the normal bundle of S.

Remark 4.1.9. As you can imagine, if a submanifold S has first order extendable tangent bundle, it is likely that every foliation on S extends to a foliation of the first infinitesimal neighborhood. A result in this direction can be found in Corollary 4.4.5.

4.2 Singular foliations of infinitesimal neighborhoods

This section is devoted to adapt a couple of definitions given in Section 1.8 to the case of foliations of infinitesimal neighborhoods.

Definition 4.2.1. Let M be a complex manifold of dimension n and S be a complex submanifold of codimension m. A **singular foliation** of $S(k)$ is a rank l (with $l \leq n-m$) coherent subsheaf $\mathcal{F}$ of $\mathcal{T}_{S(k)}$, such that:

- for every $x \in S$ we have that $[\mathcal{F}_x, \mathcal{F}_x] \subseteq \mathcal{F}_x$ (where the bracket is the one defined in Lemma 3.3.4);
- the restriction of $\mathcal{F}$ to S, denoted by $\mathcal{F}\,|_S$, is a rank l singular foliation of S.

Definition 4.2.2. Let $\mathcal{F}$ be a singular holomorphic foliation of $S(k)$. We set $\mathcal{N}_{\mathcal{F}} = \mathcal{T}_{S(k)} \,/\, \mathcal{F}$ and we denote by $S(\mathcal{F}) := \mathrm{Sing}(\mathcal{N}_{\mathcal{F}})$ the **singular set of the foliation**.

Definition 4.2.3. Let $\mathcal{F}$ be a singular foliation of $S(k)$. We say $\mathcal{F}$ is **reduced** if it is full in $\mathcal{T}_{S(k)}$, i.e., for any open set U in S we have that

$$\Gamma(U, \mathcal{T}_{S(k)}) \cap \Gamma(U \setminus S(\mathcal{F}), \mathcal{F}) = \Gamma(U, \mathcal{F}).$$

Lemma 4.2.4. *Let S be a submanifold of M and let $\mathcal{F}$ be a singular foliation of $S(k)$; then there exists a canonical way to associate to it a reduced singular foliation of $S(k)$.*

Proof. We cover now a neighborhood of S by open sets $\{U_\alpha\}$ such that $\mathcal{F}_{U_\alpha \cap S}$ is generated by $v_{1,\alpha}, \ldots, v_{l,\alpha}$ and on each U_α we can extend the $v_{i,\alpha}$ to logarithmic vector fields $\tilde{v}_{i,\alpha}$ on U_α. On U_α the $\tilde{v}_{i,\alpha}$ define a distribution with sheaf of sections $\mathcal{D}_\alpha$; please remark that this is a sheaf on U_α, not on the whole M. We define $\mathcal{N}_{\mathcal{D}_\alpha} = \mathcal{T}_M\,|_{U_\alpha} / \,\mathcal{D}_\alpha$ and denote by $S(\mathcal{D}_\alpha)$ the set of singularity of $\mathcal{N}_{\mathcal{D}_\alpha}$. In general, this distribution may not be reduced, i.e. $\Gamma(U_\alpha, \mathcal{T}_M) \cap \Gamma(U_\alpha \setminus S(D_\alpha), \mathcal{D}_\alpha) \neq \Gamma(U_\alpha, \mathcal{D}_\alpha)$. We take now the annihilator $(\mathcal{D}_\alpha)^a = \{\omega \in \Omega_M \mid \omega(v) = 0 \text{ for every } v \in \mathcal{D}_\alpha\}$. If we take its annihilator $\tilde{\mathcal{D}}_\alpha := ((\mathcal{D}_\alpha)^a)^a = \{w \in \mathcal{T}_M \mid \omega(w) = 0 \text{ for every } \omega \in (\mathcal{D}_\alpha)^a\}$ we get now a reduced sheaf of sections of the distribution, generated by sections $\tilde{w}_{1,\alpha}, \ldots, \tilde{w}_{l,\alpha}$; we can take the same l because, since we are dealing with coherent sheaves, the rank is constant outside the singularity set.

Since $\Gamma(U_\alpha, \mathcal{D}_\alpha) \subset \Gamma(U_\alpha, \tilde{\mathcal{D}}_\alpha)$ we have that, $\tilde{v}_{i,\alpha} = (h_\alpha)_i^j \tilde{w}_{j,\alpha}$, where $(h_\alpha)_i^j$ is a matrix of holomorphic functions that **may** be singular on a subset of U_α of codimension smaller than 2, contained in $S(\mathcal{D}_\alpha)$. Remark also that $S(\mathcal{F}) \subset S(\mathcal{D}_\alpha)$ and that the $\tilde{w}_{i,\alpha}$ are logarithmic vector fields.

We want to check now that $\tilde{\mathcal{D}}_\alpha \otimes \mathcal{O}_{S(k)}\,|_{(U_\alpha \cap S) \setminus S(\mathcal{F})}$ generates $\mathcal{F}$ and is involutive. We will denote the restriction of $\tilde{w}_{i,\alpha}$ to the k-th infinitesimal neighborhood by $w_{i,\alpha}$. Indeed, outside the singularity set, the matrix $(h_\alpha)_i^j$ is invertible as a matrix of holomorphic functions, with inverse $(g_\alpha)_j^i$ which implies that the

$w_{i,\alpha}$'s generate $\mathcal{F}$. We check the involutivity:

$$\begin{aligned}
[\tilde{w}_{i,\alpha}, \tilde{w}_{i',\alpha}] \otimes [1]_{k+1} =& [(g_\alpha)_i^j \tilde{v}_{j,\alpha}, (g_\alpha)_{i'}^{j'} \tilde{v}_{j',\alpha}] \otimes [1]_{k+1} \\
=& [(g_\alpha)_i^j]_{k+1} v_{j,\alpha}([(g_\alpha)_{i'}^{j'})]_{k+1}) v_{j',\alpha} \\
& - [(g_\alpha)_{i'}^{j'}]_{k+1} v_{j'\alpha}([(g_\alpha)_i^j]_{k+1}) v_{j,\alpha} \\
& + [(g_\alpha)_{i'}^{j'}]_{k+1}[(g_\alpha)_i^j]_{k+1}[v_{j,\alpha}, v_{j',\alpha}].
\end{aligned}$$

Remark that $(g_\alpha)_i^j$ is a matrix of meromorphic functions on U_α (this follows from the Cramer rule for the inverse of a matrix), and its inverse is a matrix of holomorphic functions. Now, for each $v_{j,\alpha}$ we have that

$$v_{j,\alpha}([(g_\alpha)_{i'}^{j'}]_{k+1}) = -[(g_\alpha)_{i'}^{j''}]_{k+1} v_{j,\alpha}([(h_\alpha)_{j''}^{i''}]_{k+1})[(g_\alpha)_{i''}^{j'}]_{k+1},$$

and therefore:

$$\begin{aligned}
&[(g_\alpha)_i^j]_{k+1} v_{j,\alpha}([(g_\alpha)_{i'}^{j'})]_{k+1}) v_{j',\alpha} \\
&\qquad = - [(g_\alpha)_i^j]_{k+1}[(g_\alpha)_{i'}^{j''}]_{k+1} v_{j,\alpha}([(h_\alpha)_{j''}^{i''}]_{k+1})[(g_\alpha)_{i''}^{j'}]_{k+1} v_{j',\alpha} \\
&\qquad = - [(g_\alpha)_{i'}^{j''}]_{k+1} w_{i,\alpha}([(h_\alpha)_{j''}^{i''}]_{k+1}) w_{i'',\alpha}.
\end{aligned}$$

A similar reasoning holds for the second summand in the involutivity check. If we denote by $[a_{j,j'}^{j''}]_{k+1}$ the elements of $\mathcal{O}_{S(k)}$ such that

$$[v_{j,\alpha}, v_{j',\alpha}] = a_{j,j'}^{j''} v_{j'',\alpha}$$

we have that:

$$\begin{aligned}
&[(g_\alpha)_{i'}^{j'}]_{k+1}[(g_\alpha)_i^j]_{k+1}[v_{j,\alpha}, v_{j',\alpha}] \\
&\qquad\qquad = - [(g_\alpha)_{i'}^{j'}]_{k+1}[(g_\alpha)_i^j]_{k+1}[a_{j,j'}^{j''}]_{k+1} v_{j'',\alpha} \\
&\qquad\qquad = - [(g_\alpha)_{i'}^{j'}]_{k+1}[(g_\alpha)_i^j]_{k+1}[a_{j,j'}^{j''}]_{k+1}[(h_\alpha)_{j''}^{i''}]_{k+1} w_{i'',\alpha}.
\end{aligned}$$

The point these computations prove is that $[\tilde{w}_{i,\alpha}, \tilde{w}_{i',\alpha}] \otimes [1]_{k+1}$ belongs to the module generated by the $w_{i,\alpha}$'s over the meromorphic functions. But, a priori, we know that this bracket is a holomorphic section of $\mathcal{T}_{S(1)}$ and therefore it belongs to the $\mathcal{O}_{S(1)}$-module generated by the $w_{i,\alpha}$'s. □

Remark 4.2.5. By the proof of the Lemma above and by 1.8.15 we have an important consequence: each one of the extensions $\tilde{w}_{i,\alpha}$ has a singularity set of codimension at least 2.

4.3 2-splittings submanifolds

In this section we will give some new insight on the notion of 2-splitting (Definition 1.10); many of the results in this section are strictly connected with those in [4, 5]. The main idea is that, given a 2-splitting of a submanifold, there exist maps which permit us to project vector fields transversal to the first infinitesimal neighborhood into vector fields which are tangential to the first infinitesimal neighborhood. Recall that when we have a splitting submanifold, we have a lot of really interesting properties; recalling Proposition 1.10.5, the following definition makes sense.

Definition 4.3.1. Let S be a submanifold splitting into M; if $\mathcal{F}$ is foliation of M of rank l strictly smaller than the dimension of S, if we denote by σ^* the map from $\mathcal{T}_{M,S}$ to $\mathcal{T}_S$ given in Proposition 1.10.5, we shall denote by $\mathcal{F}^\sigma$ the coherent sheaf of $\mathcal{O}_S$-modules given by

$$\mathcal{F}^\sigma := \sigma^*(\mathcal{F}|_S).$$

We shall say that σ^* is **$\mathcal{F}$-faithful outside an analytic subset** $\Sigma \subset S$ if $\mathcal{F}^\sigma$ is a regular foliation of S of rank l on $S \setminus \Sigma$. If $\Sigma = \emptyset$ we shall simply say that σ^* is **$\mathcal{F}$-faithful**.

We refer to [4] for a treatment of $\mathcal{F}$-faithfulness in the case of splittings. Assume that σ^* is $\mathcal{F}$-faithful: an interesting question is whether there exists a sufficient condition for $\mathcal{F}^\sigma$ to extend to the first infinitesimal neighborhood. A simple way to obtain such an extension would be to find an analogue of σ^* from $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$, which restricted to $\mathcal{T}_{M,S}$ coincides with σ^*. First of all, we can suppose we are working on a splitting submanifold S. Then, we look for a splitting of the following sequence:

$$0 \to \mathcal{T}_{S(1)} \to \mathcal{T}_{M,S(1)} \to \mathcal{N}_{S(1)} \to 0, \tag{4.2}$$

where $\mathcal{N}_S(1)$ is the quotient of the two modules.

Remark 4.3.2. Let $(U_\alpha, z_\alpha^1, \dots, z_\alpha^n)$ be a coordinate system adapted to S. Please remember Remark 3.3.2; since S is a submanifold the ideal of S is generated by $z_\alpha^1, \dots, z_\alpha^r$ and we have that $\mathcal{T}_{S(1)}$ is generated in U_α by:

$$[z_\alpha^r]_2 \frac{\partial}{\partial z_\alpha^s}, \frac{\partial}{\partial z_\alpha^{m+1}}, \dots, \frac{\partial}{\partial z_\alpha^n},$$

while $\mathcal{T}_{M,S(1)}$ is generated on U_α by

$$\frac{\partial}{\partial z_\alpha^1}, \dots, \frac{\partial}{\partial z_\alpha^n}.$$

Let $\partial_{r,\alpha}$ be the image of $\partial/\partial z_\alpha^r$ in $\mathcal{N}_{S(1)}$ and ω_α^r its dual element. Now let

$$v = [f_\alpha^k]_2 \frac{\partial}{\partial z_\alpha^k}$$

be a section of $\mathcal{T}_{M,S(1)}$; we can see that the image of v in $\mathcal{N}_{S(1)}$ is nothing else than $[f_\alpha^r]_1 \partial_{r,\alpha}$. We denote by $[v]$ the equivalence class of v in $\mathcal{N}_{S(1)}$; please note that, given a function $[g]_2$ in $\mathcal{O}_{S(1)}$ the $\mathcal{O}_{S(1)}$-module structure is given by

$$[g]_2 \cdot [v] = [\theta_1([g]_2) \cdot v].$$

We compute now the transition functions of $\mathcal{N}_{S(1)}$; if we are in an atlas adapted to S we have that $z_\alpha^s = h_{\alpha\beta,r}^s z_\beta^r$. We have that:

$$\begin{aligned}
\partial_{r,\alpha} &= \left[\frac{\partial}{\partial z_\alpha^r}\right] = \left[\frac{\partial z_\beta^k}{\partial z_\alpha^r}\frac{\partial}{\partial z_\beta^k}\right] = \left[\frac{\partial z_\beta^s}{\partial z_\alpha^r}\frac{\partial}{\partial z_\beta^s}\right] = \left[\frac{\partial(h_{\alpha\beta,r'}^s z_\beta^{r'})}{\partial z_\alpha^r}\frac{\partial}{\partial z_\beta^s}\right] \\
&= \left[\frac{\partial(h_{\alpha\beta,r'}^s)}{\partial z_\alpha^r} z_\beta^{r'} \frac{\partial}{\partial z_\beta^s}\right] + \left[h_{\alpha\beta,r'}^s \delta_r^{r'} \frac{\partial}{\partial z_\beta^s}\right] = [h_{\alpha\beta,r}^s]_2 \partial_{s,\beta},
\end{aligned} \tag{4.3}$$

where last equality comes from the equivalence relations that define $\mathcal{N}_{S(1)}$.

Remark 4.3.3. Please note that the transition functions for $(\mathcal{N}_{S(1)})^*$, as an $\mathcal{O}_{S(1)}$-module are given by:

$$\omega^s_\beta = [h^s_{\alpha\beta,r}]_2 \omega^r_\alpha.$$

Please note also that $(\mathcal{N}_{S(1)})^*$ is isomorphic to $\mathcal{I}_S \,/\, \mathcal{I}_S{}^2$ with the structure of $\mathcal{O}_{S(1)}$ module given by the projection $\theta_1 : \mathcal{O}_{S(1)} \to \mathcal{O}_S$.

Lemma 4.3.4. *Let M be a n-dimensional complex manifold, S a submanifold of codimension r. Sequence* (4.2) *splits if S is* 2*-splitting, i.e., if there exists an atlas adapted to S such that:*

$$\left[\frac{\partial z^p_\beta}{\partial z^r_\alpha}\right]_2 \equiv [0]_2, \tag{4.4}$$

for $p = m+1, \ldots, n$ and $r = 1, \ldots, m$.

Proof. We have to compute the image in $H^1(S, \mathrm{Hom}(\mathcal{N}_{S(1)}, \mathcal{T}_{S(1)})$ through the coboundary operator of the class $\{U_\alpha \cap S, \omega^r_\alpha \otimes \partial_{r,\alpha}\}$ (i.e., the identity) in $H^0(\mathcal{U}, \mathrm{Hom}(\mathcal{N}_{S(1)}, \mathcal{N}_{S(1)}))$. We compute then:

$$\begin{aligned}
\delta(U_\alpha, \omega^r_\alpha \otimes \partial_{r,\alpha}) =& \omega^r_\beta \otimes \frac{\partial}{\partial z^r_\beta} - \omega^s_\alpha \otimes \frac{\partial}{\partial z^s_\alpha} \\
=& \omega^r_\beta \otimes \frac{\partial}{\partial z^r_\beta} - \left[\frac{\partial z^s_\alpha}{\partial z^{r'}_\beta}\frac{\partial z^k_\beta}{\partial z^s_\alpha}\right]_2 \omega^{r'}_\beta \otimes \frac{\partial}{\partial z^k_\beta} \\
=& \omega^r_\beta \otimes \frac{\partial}{\partial z^r_\beta} - \left[\frac{\partial z^s_\alpha}{\partial z^{r'}_\beta}\frac{\partial z^r_\beta}{\partial z^s_\alpha}\right]_2 \omega^{r'}_\beta \otimes \frac{\partial}{\partial z^r_\beta} \\
&- \left[\frac{\partial z^p_\beta}{\partial z^s_\alpha}\frac{\partial z^s_\alpha}{\partial z^{r'}_\beta}\right]_2 \omega^{r'}_\beta \otimes \frac{\partial}{\partial z^p_\beta} \\
=& - \left[\frac{\partial z^p_\beta}{\partial z^s_\alpha}\frac{\partial z^s_\alpha}{\partial z^{r'}_\beta}\right]_2 \omega^{r'}_\beta \otimes \frac{\partial}{\partial z^p_\beta}.
\end{aligned} \tag{4.5}$$

This class is clearly zero if we are using a 2-splitting atlas. □

Remark 4.3.5. In the last equality of the computation above there is marginal subtle point. We are using the fact that S is splitting. We saw in Section 1.10 that this implies that in an atlas adapted to S and to the splitting

$$\frac{\partial z^p_\alpha}{\partial z^r_\beta} \in \mathcal{I}_S, \quad \frac{\partial z^r_\alpha}{\partial z^p_\beta} \in \mathcal{I}_S\,.$$

We know also that:

$$\frac{\partial z^k_\alpha}{\partial z^r_\beta}\frac{\partial z^s_\beta}{\partial z^k_\alpha} = \delta^s_r.$$

Restricting ourselves to the 1-st infinitesimal neighborhood we have that:

$$[\delta^s_r]_2 = \left[\frac{\partial z^k_\alpha}{\partial z^r_\beta}\frac{\partial z^s_\beta}{\partial z^k_\alpha}\right]_2 = \left[\frac{\partial z^{r'}_\alpha}{\partial z^r_\beta}\frac{\partial z^s_\beta}{\partial z^{r'}_\alpha}\right]_2 + \left[\frac{\partial z^p_\alpha}{\partial z^r_\beta}\frac{\partial z^s_\beta}{\partial z^p_\alpha}\right]_2 = \left[\frac{\partial z^{r'}_\alpha}{\partial z^r_\beta}\frac{\partial z^s_\beta}{\partial z^{r'}_\alpha}\right]_2,$$

using the splitting hypothesis.

Remark 4.3.6. Looking at how we have constructed the splitting in the former Lemma, if $\tilde{\rho}$ is the θ_1-derivation associated to the splitting of S, we have that the splitting morphism $\sigma^* : \mathcal{T}_{M,S(1)} \to \mathcal{T}_{S(1)}$ is given in an atlas adapted to the 2-splitting by:

$$f^k \frac{\partial}{\partial z^k} \mapsto \tilde{\rho}(f^r) \frac{\partial}{\partial z^r} + f^p \frac{\partial}{\partial z^p}.$$

Now the question is under which conditions the splitting of sequence (4.2) is equivalent to the existence of a 2-splitting atlas. It seems like the splitting of this sequence is not enough. Indeed, if we try to follow the usual approach in proving the argument, as in Proposition 1.10.8, we have some problems. The first thing we can remark is that the dual of $\mathcal{T}_{M,S(1)}$ is nothing else than $\Omega_M \otimes \mathcal{O}_{S(1)}$. Now, a splitting of (4.2) implies there exists a map γ from $\Omega_M \otimes \mathcal{O}_{S(1)}$ to $(\mathcal{N}_{S(1)})^*$ and since we remarked that $(\mathcal{N}_{S(1)})^*$ is isomorphic to $\mathcal{I}_S / {\mathcal{I}_S}^2$ as an $\mathcal{O}_{S(1)}$-module, through a map

$$\tau : (\mathcal{N}_{S(1)})^* \to \mathcal{I}_S / {\mathcal{I}_S}^2 .$$

this gives rise to a splitting of the map

$$d_2 : \mathcal{I}_S / {\mathcal{I}_S}^2 \to \Omega_M \otimes \mathcal{O}_{S(1)}$$

which sends an element $[f]_2$ of $\mathcal{I}_S / {\mathcal{I}_S}^2$ into $df \otimes [1]_2$. Now, there exists a well defined map d_3 from $\mathcal{O}_M / {\mathcal{I}_S}^3$ to $\Omega_M \otimes \mathcal{O}_{S(1)}$, which sends a class $[f]_3$ to $d\tilde{f} \otimes [1]_2$. The big problem is that, even if we suppose S comfortably embedded, the splitting of (4.2) only gives us a map between $\mathcal{O}_{S(1)}$ and the image through the splitting $\nu : \mathcal{I}_S / {\mathcal{I}_S}^3 \to \mathcal{I}_S / {\mathcal{I}_S}^2$ of $\mathcal{I}_S / {\mathcal{I}_S}^2$ in $\mathcal{I}_S / {\mathcal{I}_S}^3$ and this map is not surjective. Therefore it is not a $\theta_{2,1}$-derivation splitting the short exact sequence of morphisms of rings

$$0 \longrightarrow \mathcal{I}_S / {\mathcal{I}_S}^3 \longrightarrow \mathcal{O}_{S(1)} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Remark 4.3.7. To solve this problem we could find under which conditions there exists a splitting of the map $\tilde{d}_3 : {\mathcal{I}_S}^2 / {\mathcal{I}_S}^3 \to \Omega_{M,S(1)}$. Using such a splitting and the comfortable embedding we could find a $\theta_{2,1}$-derivation splitting $\iota : \mathcal{I}_S / {\mathcal{I}_S}^3 \to \mathcal{O}_{S(1)}$.

Definition 4.3.8. If $\mathcal{F}$ is foliation of M of rank l strictly smaller than dimension S, if we denote by σ_2^* the map from $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$ splitting sequence (4.2), we shall denote by $\mathcal{F}^{\sigma_2}$ the coherent sheaf of $\mathcal{O}_S(1)$-modules given by

$$\mathcal{F}^{\sigma_2} := \sigma_2^*(\mathcal{F}|_{S(1)}).$$

We shall say that σ_2^* is **first order $\mathcal{F}$-faithful outside an analytic subset** $\Sigma \subset S$ if $\mathcal{F}^{\sigma_2}$ is a regular foliation of $S(1)$ of rank l on $S \setminus \Sigma$. If $\Sigma = \emptyset$ we shall simply say that σ_2^* is **first order $\mathcal{F}$-faithful**.

We state some results coming from [4], in particular Lemma 7.5 and Lemma 7.6 about $\mathcal{F}$-faithfulness.

Lemma 4.3.9 ([4], Lemma 7.5)**.** *Let S be a splitting submanifold of a complex manifold M , and let $\mathcal{F}$ be a holomorphic foliation on M of dimension equal to 1 or to the dimension of S. If there exists $x_0 \in S \setminus S(\mathcal{F})$ such that $\mathcal{F}$ is tangent to S at x_0 , i.e., $(\mathcal{F}|_S)x_0 \subset \mathcal{T}_{S,x_0}$, then any splitting morphism is $\mathcal{F}$-faithful outside a suitable analytic subset of S.*

Lemma 4.3.10 ([4], Lemma 7.6)**.** *Let S be a non-singular hypersurface splitting in a complex manifold M , and let $\mathcal{F}$ be a one dimensional holomorphic foliation on M. Assume that S is not contained in $S(\mathcal{F})$. Then there is at most one splitting morphism σ^* which is not $\mathcal{F}$-faithful outside a suitable analytic subset of S.*

Indeed, speaking of first order $\mathcal{F}$-faithfulness we have a simple results which gives us some insight.

Lemma 4.3.11. *Let S be a submanifold 2-splitting in a complex manifold M , and let $\mathcal{F}$ be a one dimensional holomorphic foliation on M. Let $\sigma_2^* : \mathcal{T}_{M,S(1)} \to \mathcal{T}_{S(1)}$ be the splitting morphism. If $\sigma_2^*|_S$ is $\mathcal{F}$-faithful outside an analytic subset Σ, then σ_2^* is first order $\mathcal{F}$-faithful outside Σ.*

Proof. We check that $\mathcal{F}^{\sigma_2}$ satisfies the requests of Definition 4.1.1. By hypothesis $\mathcal{F}^{\sigma_2}|_S$ is a foliation of S. Since the rank of $\mathcal{F}^{\sigma_2}$ is 1 it is an involutive subbundle of $\mathcal{T}_{S(1)}$; moreover, for each point $x \in S \setminus \Sigma$ we can find a generator v of $\mathcal{F}^{\sigma_2}_x$ such that $v|_S$ is non zero. Therefore, $\mathcal{T}_{S(1),x} \,/\, \mathcal{F}^{\sigma_2}_x$ is $\mathcal{O}_{S(1)}$-free. □

Directly from this last Lemma and Lemma 4.3.9 we have that.

Corollary 4.3.12. *Let S be a splitting submanifold of a complex manifold M, and let $\mathcal{F}$ be a holomorphic foliation on M of dimension equal to 1 or to the dimension of S. If there exists $x_0 \in S \setminus Sing(F)$ such that $\mathcal{F}$ is tangent to S at x_0, i.e., $(\mathcal{F}|_S)x_0 \subset \mathcal{T}_{S,x_0}$, then any 2-splitting morphism is first order $\mathcal{F}$-faithful outside a suitable analytic subset of S.*

4.4 Extension of foliations

Let S be a codimension m submanifold of an n-dimensional complex manifold M and let $\mathcal{F}$ be a foliation of S. Thanks to the Holomorphic Frobenius theorem, we know that there exists an atlas $\{(U_\alpha; z^1_\alpha, \dots, z^n_\alpha)\}$ adapted to S and $\mathcal{F}$. In such an atlas we know that $\mathcal{F} = \ker(dz^{m+l+1}_\alpha|_S, \dots, dz^n_\alpha|_S)$. An equivalent formulation of the Frobenius theorem states that a submodule of $\Omega^1(S)$ is integrable if and only if each stalk is generated by exact forms.

We denote by $\pi : N_S \to S$ the normal bundle of S; the map π is holomorphic and therefore $\pi^*(dz^k_\alpha|_S)$ is a well defined local holomorphic 1-form on $\pi^{-1}(U_\alpha) \subset N_S$. Moreover, since $\{dz^{m+l+1}_\alpha|_S, \dots, dz^n_\alpha|_S\}$ is an integrable system of 1-forms, so is $\{\pi^*(dz^{m+l+1}_\alpha|_S), \dots, \pi^*(dz^n_\alpha|_S)\}$. Then $\{\pi^*(dz^{m+l+1}_\alpha|_S), \dots, \pi^*(dz^n_\alpha|_S)\}$ defines a foliation $\tilde{\mathcal{F}}$ of N_S, whose leaves are the preimages of the leaves of $\mathcal{F}$ through π.

Since S is regular TM is trivialized on each coordinate neighborhoods and so is N_S. In what follows we use the atlas $\{(\pi^{-1}(U_\alpha), v^1_\alpha, \dots, v^m_\alpha, z^{m+1}_\alpha, \dots, z^n_\alpha\}$ of N_S given by the trivializations of the normal bundle, where v^r_α are the coordinates in the fiber; then $\tilde{\mathcal{F}}$ is generated on $\pi^{-1}(U_\alpha)$ by

$$\frac{\partial}{\partial v^1_\alpha}, \dots, \frac{\partial}{\partial v^m_\alpha}, \left.\frac{\partial}{\partial z^{m+1}_\alpha}\right|_S, \dots, \left.\frac{\partial}{\partial z^{m+l}_\alpha}\right|_S.$$

The fibers of π are the leaves of a holomorphic foliation of N_S, called the **vertical foliation**, which we denote by $\mathcal{V}$. On $\pi^{-1}(U_\alpha)$ it is generated by

$$\frac{\partial}{\partial v^1_\alpha}, \dots, \frac{\partial}{\partial v^m_\alpha}.$$

We study now the splitting of the following sequence, when restricted to the first infinitesimal neighborhood of the embedding of S as the zero section of N_S:

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{\text{pr}} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0\ . \tag{4.6}$$

The result of Section 1.9 tells us that the splitting of the sequence is equivalent to the vanishing of a cohomology class in $H^1(M, \mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \mathcal{V}))$ and that the splitting of this sequence is equivalent to the fact that there exist an isomorphism $\tilde{\mathcal{F}} \simeq \mathcal{V} \oplus \tilde{\mathcal{F}}/\mathcal{V}$ compatible with the projection pr and the map ι.

Remark 4.4.1. Please note that $\tilde{\mathcal{F}}/\mathcal{V}$, when restricted to S is nothing else but the foliation $\mathcal{F}$. This follows directly from our construction of $\tilde{\mathcal{F}}$ as the pull-back foliation defined by the integrable system $\{\pi^*(dz_\alpha^{m+l+1}|_S), \ldots, \pi^*(dz_\alpha^n|_S)\}$.

Lemma 4.4.2. *Let S be a splitting submanifold in M. If there exists a foliation of the first infinitesimal neighborhood of the embedding of S as the zero section of its normal bundle, then there exists a foliation of the first infinitesimal neighborhood of S embedded in M.*

Proof. If there exists a foliation of the first infinitesimal neighborhood of the embedding of S as the zero section of its normal bundle we can find an atlas $\{V_\alpha, v_\alpha^1, \ldots, v_\alpha^m, z_\alpha^{m+1}, \ldots, z_\alpha^n\}$ of N_S such that, if $V_\alpha \cap V_\beta \cap S \neq \emptyset$ we have that

$$\left[\frac{\partial v_\alpha^r}{\partial z_\beta^i}\right]_2 = [0]_2; \quad \frac{\partial z_\alpha^{t'}}{\partial z_\beta^i} = 0$$

where $r = 1, \ldots, m$ and $t' = m + l + 1, \ldots, n$ (we are *not* following our usual convention).

We use the isomorphism $\phi : \mathcal{O}_{N_S}/\mathcal{I}^2_{S,N_S} \to \mathcal{O}_M/\mathcal{I}^2_S$, taking the images

$$[\tilde{z}_\alpha^1]_2 = \phi([v_\alpha^1]_2), \ldots, [\tilde{z}_\alpha^r]_2 = \phi([v_\alpha^r]_2), [\tilde{z}_\alpha^{m+1}]_2 = \phi([z_\alpha^{m+1}]_2), \ldots, [\tilde{z}_\alpha^n]_2 = \phi(z_\alpha^n);$$

there exists $U_\alpha \supset \pi(V_\alpha)$ (modulo shrinking) where we can choose representatives of these classes such that $(U_\alpha, \tilde{z}_\alpha^1, \ldots, \tilde{z}_\alpha^n)$, is a coordinate system adapted to S and $\mathcal{F}|_S$. If $U_\alpha \cap U_\beta \cap S \neq \emptyset$ we can check that, since $\partial/\partial \tilde{z}_\beta^{m+1}, \ldots, \partial/\partial \tilde{z}_\alpha^{m+l}$ are logarithmic

$$\frac{\partial [\tilde{z}_\beta^r]_2}{\partial \tilde{z}_\alpha^i} = \left[\frac{\partial \tilde{z}_\beta^r}{\partial \tilde{z}_\alpha^i}\right]_2 = \left[\frac{\partial u_\beta^r}{\partial z_\alpha^i}\right]_2 = [0]_2,$$

for $r = 1, \ldots, m$ and $i = m + 1, \ldots, m + l$. Following the same line of thought

$$\frac{\partial [\tilde{z}_\beta^t]_2}{\partial \tilde{z}_\alpha^i} = \left[\frac{\partial \tilde{z}_\beta^t}{\partial \tilde{z}_\alpha^i}\right]_2 = \left[\frac{\partial z_\beta^t}{\partial z_\alpha^i}\right]_2 = [0]_2,$$

for $t = m + l + 1, \ldots, n$ and $i = m + 1, \ldots, m + l$. □

So, the problem of extending a foliation boils down, in the splitting case, to understand when (4.6) splits and the image through the splitting of $\mathcal{F}/\mathcal{V}$ is involutive. We start by finding a sufficient condition for the splitting of the sequence.

Proposition 4.4.3. *Let M be a complex manifold of dimension n, and S a splitting codimension m submanifold. Let $\mathcal{F}$ be a foliation of S and $\pi : N_S \to M$ the normal bundle of S in M. Let $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ and $\mathcal{V}$ the vertical foliation given by* $\ker d\pi$*. The sequence:*

$$0 \longrightarrow \mathcal{V} \xrightarrow{\ \iota\ } \tilde{\mathcal{F}} \xrightarrow{\ pr\ } \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits if there exists an atlas adapted to $\mathcal{F}$ and S such that

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^i \partial z_\beta^s} \in \mathcal{I}_S,$$

for all $r, s = 1, \dots, m$ and $i = m+1, \dots, m+l$.

Proof. We compute the obstruction to the splitting of the sequence, following Section 1.9: we apply the functor $\mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \cdot)$ to sequence (4.6) and compute the image of the identity through the coboundary map

$$\delta^* : H^0(S, \mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \tilde{\mathcal{F}}/\mathcal{V})) \to H^1(S, \mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \mathcal{V})).$$

We fix an atlas $\{U_\alpha, z_\alpha\}$ adapted to S and $\mathcal{F}$ and we denote the quotient frame for $\tilde{\mathcal{F}}/\mathcal{V}$ by $\{\partial_{m+1,\alpha}, \dots, \partial_{m+l,\alpha}\}$ (i.e., $\partial_{m+1,\alpha}$ is the equivalence class of $\partial/\partial z_\alpha^{m+1}|_S$) and by $\{\omega_\alpha^{m+1}, \dots, \omega_\alpha^{m+l}\}$ its dual frame. The cocycle representing the identity in $H^0(S, \mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \tilde{\mathcal{F}}/\mathcal{V}))$ is then represented as $\{U_\alpha, \omega_\alpha^j \otimes \partial_{j,\alpha}\}$; the obstruction to the splitting of the sequence is then

$$\begin{aligned}
\delta^*\{\omega_\alpha^j \otimes \partial_{j,\alpha}\} &= \omega_\beta^j \otimes \frac{\partial}{\partial z_\alpha^j} - \omega_\alpha^j \otimes \frac{\partial}{\partial z_\alpha^j} \\
&= \omega_\beta^j \otimes \frac{\partial}{\partial z_\alpha^j} - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial z_\beta^{j'}}{\partial z_\alpha^j}\right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial z_\beta^{j'}} - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial v_\beta^r}{\partial z_\alpha^j}\right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r} \\
&= -\left[\frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial v_\beta^r}{\partial z_\alpha^j}\right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r}. \qquad (4.7)
\end{aligned}$$

The vanishing of (4.7) is a sufficient condition for the splitting of the sequence; this class vanishes if $\partial v_\beta^r / \partial z_\alpha^j$ belong to $\mathcal{I}_{N_S}^2$. Moreover, the coordinate changes map of N_S have a peculiar structure:

$$v_\beta^r = v_\alpha^s \frac{\partial z_\beta^r}{\partial z_\alpha^s}.$$

Therefore:

$$\begin{aligned}
-\left[\frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial v_\beta^r}{\partial z_\alpha^j}\right]_2 &= -\left[v_\alpha^s \frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial}{\partial z_\alpha^j}\left(\frac{\partial z_\beta^r}{\partial z_\alpha^s}\right)\right]_2 \\
&= -\left[v_\alpha^s \frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial^2 z_\beta^r}{\partial z_\alpha^j \partial z_\alpha^s}\right]_2; \qquad (4.8)
\end{aligned}$$

using the isomorphism between $\mathcal{I}_S/\mathcal{I}_S^2$ and $\mathcal{I}_{S,N_S}/\mathcal{I}_{S,N_S}^2$ we see that the last expression vanishes if

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^i \partial z_\beta^s} \in \mathcal{I}_S,$$

for all $r, s = 1, \dots, m$ and $i = m+1, \dots, m+l$. ∎

Remark 4.4.4. Since we are working in an atlas of N_S adapted to S and $\mathcal{F}$ we have that

$$\frac{\partial z_\alpha^p}{\partial v_\beta^r} \equiv 0 \quad , \quad \frac{\partial z_\alpha^t}{\partial z_\beta^i} \equiv 0,$$

for $p = m+1, \ldots, n$, $r = 1, \ldots, m$, $t = m+l+1, \ldots, n$, $i = m+1, \ldots, n$ (please remark that we are *not* following our usual convention). Looking at the transition functions of the tangent bundle of N_S, in an atlas adapted to S and $\mathcal{F}$, on the first infinitesimal neighborhood of the embedding of S as the zero section of N_S, the following equality holds:

$$[\delta_j^i]_2 = \left[\frac{\partial v_\alpha^r}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial v_\alpha^r} + \frac{\partial z_\alpha^p}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^p} \right]_2,$$

now, since $\partial z_\alpha^p / \partial v_\beta^r \equiv 0$ we have that

$$[\delta_j^i]_2 = \left[\frac{\partial z_\alpha^p}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^p} \right]_2$$

and since $\partial z_\alpha^t / \partial z_\beta^i \equiv 0$ for $t = m+l+1, \ldots, n$, $i = m+1, \ldots, n$ we have that

$$[\delta_j^i]_2 = \left[\frac{\partial z_\alpha^{i'}}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^{i'}} \right]_2,$$

where $i, j = m+1, \ldots, m+l$ and we sum over $i' = m+1, \ldots, m+l$.

Corollary 4.4.5. *Let M be a n-dimensional complex manifold, S a codimension m splitting submanifold, $\mathcal{F}$ a regular foliation of S. Suppose S admits first order extendable tangent bundle, then $\mathcal{F}$ extends to a subbundle of $\mathcal{T}_{S(1)}$.*

Proof. We work in an atlas adapted to S and $\mathcal{F}$; by Corollary 4.1.4 first order extendable tangent bundle implies

$$\left[\frac{\partial z_\alpha^r}{\partial z_\beta^p} \right]_2 = 0$$

which in turn implies

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^s \partial z_\beta^p} \in \mathcal{I}_S$$

and therefore that

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^s \partial z_\beta^i} \in \mathcal{I}_S,$$

for $r, s = 1, \ldots, m$, $i = m+1, \ldots, m+l$, which in turn implies the vanishing of (4.7) and the splitting of sequence (4.6); the extension of $\mathcal{F}$ is then given by the image of $\tilde{\mathcal{F}}/\mathcal{V}$ in $\tilde{\mathcal{F}}$. $\square$

Remark 4.4.6. Please remark that the extension of $\mathcal{F}$ as a subsheaf of $\mathcal{T}_{S(1)}$ may not be involutive.

Lemma 4.4.7. *Let S be a splitting submanifold of M, let $\mathcal{F}$ be foliation of S; if the sequence*

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{pr} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0\,.$$

splits on the first infinitesimal neighborhood of S embedded as the zero section of its normal bundle in M, then $\mathcal{F}$ extends as a subsheaf of $\mathcal{T}_{S(1)}$.

Proof. On S we have the following isomorphism:

$$\tilde{\mathcal{F}}|_S = \mathcal{V}|_S \oplus \mathcal{F}\,. \tag{4.9}$$

Indeed if we are working in an atlas adapted to S and $\mathcal{F}$; this follows directly from our construction of $\tilde{\mathcal{F}}$ as the pull-back foliation defined by the integrable system $\{\pi^*(dz_\alpha^{m+l+1}|_S), \ldots, \pi^*(dz_\alpha^n|_S)\}$. Therefore we have that $\tilde{\mathcal{F}}/\mathcal{V}|_S \simeq \mathcal{F}$ and this implies that the cochain representing (4.7) vanishes identically when restricted to S. Let $\{U_\alpha, \tau_\alpha\}$ be local splittings and let v be a section of $\tilde{\mathcal{F}}/\mathcal{V}$; if the sequence splits there exists a cochain in $C^0(S, \mathrm{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \mathcal{V}))$ denoted by $\{U_\alpha, \sigma_\alpha\}$ such that

$$\sigma_\beta - \sigma_\alpha = \tau_\beta - \tau_\alpha.$$

Since the cochain representing (4.7) vanishes identically when restricted to S the components of the cochain $\{U_\alpha, \sigma_\alpha\}$ are identically 0 when restricted to S. The image $\tau_\alpha - \sigma_\alpha(v)$ is a section of $\tilde{\mathcal{F}}$. From the discussion above we can remark that $\sigma_\alpha(v)|_S \equiv 0$; moreover, since on S (4.9) holds we have that $\tau_\alpha(v)|_S \in \mathcal{F} \subset \mathcal{T}_S$ and this proves that v is a logarithmic section of $\mathcal{T}_{M,S(1)}$ and therefore belongs to $\mathcal{T}_{S(1)}$. □

Corollary 4.4.8. *Let S be a splitting submanifold of M, let $\mathcal{F}$ be a rank 1 foliation of S; if the sequence*

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{pr} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits on the first infinitesimal neighborhood of S embedded as the zero section of its normal bundle in M, then $\mathcal{F}$ extends as a foliation of the first infinitesimal neighborhood of S in M. Moreover, we can find an atlas adapted to S and $\mathcal{F}$ given by a collection of charts $\{U_\alpha, (v_\alpha^1, \ldots, v_\alpha^m, z_\alpha^{m+1} \ldots, z_\alpha^n)\}$ such that the class (4.7) can be represented by the 0 cochain.

Proof. If $\tilde{\mathcal{F}}/\mathcal{V}$ has rank 1 we have that its image through the splitting morphism of (4.6) is a rank 1 (therefore involutive) subbundle of the tangent sheaf to the first infinitesimal neighborhood of S in its normal bundle. Thanks to Lemma 4.4.2 we have the first part of the assertion. Corollary 4.1.4 gives us the second part of the assertion. □

Remark 4.4.9. The reason why the splitting of (4.6) is not a sufficient condition for the foliation to extend to the first infinitesimal neighborhood lies in the fact that the image of $\tilde{\mathcal{F}}/\mathcal{V}$ may not be involutive. If this image is involutive we have a statement similar to the one in the last Corollary; anyway even if it is not involutive, thanks to the results in section 5.2, the splitting of (4.6) is enough to get some important insights on the Khanedani-Lehmann-Suwa action.

Remark 4.4.10. We want to see what happens in coordinates when we can extend the foliation. First of all, the vanishing of the class (4.7) in cohomology means there exists a cochain $\{U_\alpha, \sigma_\alpha\} \in C^0(S_N(1), (\mathcal{F}/\mathcal{V}))^* \otimes \mathcal{V})$ such that:

$$\sigma_\beta - \sigma_\alpha = -\left[\frac{\partial z_\alpha^j}{\partial z_\beta^i}\frac{\partial v_\beta^r}{\partial z_\alpha^j}\right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r}.$$

In a coordinate system adapted to S and $\mathcal{F}$ on each U_α we can write the elements of the cochain as

$$\sigma_\alpha = [c_{j,\alpha}^s]_2 \omega_\alpha^j \otimes \frac{\partial}{\partial v_\alpha^s}.$$

Since the sequence splits when it is restricted to S we can assume that the coefficients $c_{j,\alpha}^s$ of each σ_α belong to $\mathcal{I}_S/\mathcal{I}_S^2$. Without loss of generality we can suppose the local lifts τ_α send the generators of $\tilde{\mathcal{F}}/\mathcal{V}$, that we denote by $\partial_{i,\alpha}$, in the coordinate fields $\partial/\partial z_\alpha^i$ (the difference about two different choices of lifts is absorbed by the cochain). Then a generator $\partial/\partial z_\alpha^i|_S$ of $\mathcal{F}$ on U_α extends to the section v of $\mathcal{T}_{S(1)}$ given by:

$$-[c_{j,\alpha}^s]_2 \frac{\partial}{\partial v_\alpha^s} + \frac{\partial}{\partial z_\alpha^j}.$$

Chapter 5

Partial holomorphic connections on $\mathcal{N}_{\mathcal{F},M}$

Remark 5.0.11. In this chapter we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

5.1 The action arising from a foliation of the first infinitesimal neighborhood

This is a short section showing in an explicit way how the existence of a foliation of the first infinitesimal neighborhood $\mathcal{F}$ gives rise to a partial holomorphic connection for $\mathcal{N}_{\mathcal{F},M}$ along $\mathcal{F}|_S$. The existence of a partial holomorphic connection gives rise to a splitting of the partial Atiyah sequence; the cocycle representing the cohomology class we calculated in Section 3.2 vanishes identically if we can find an atlas in the form given by Corollary 4.1.4 when we have a foliation of the first infinitesimal neighborhood. We want now to express this partial holomorphic connection explictly.

Theorem 5.1.1. *Let S be a submanifold of a complex manifold M and suppose there exists a foliation $\mathcal{F}$ of the first infinitesimal neighborhood of S. Then, there exists a flat partial holomorphic connection $(\delta, \mathcal{F})$ on $\mathcal{N}_{\mathcal{F},M}$ along $\mathcal{F}$.*

Proof. We want to define now the splitting map between $\mathcal{F}|_S$ and $\mathcal{A}_{\mathcal{N}_{\mathcal{F},M},\mathcal{F}}$; this is really simple since each of $[1]_2 \otimes \partial/\partial z^i_\alpha$ belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ (Definition 3.4.2). Therefore we define $\psi : \mathcal{F}|_S \to \mathcal{A}_{\mathcal{N}_{\mathcal{F},M},\mathcal{F}}$ as

$$\psi : \frac{\partial}{\partial z^i_\alpha} \mapsto \pi\left([1]_2 \otimes \frac{\partial}{\partial z^i_\alpha}\right),$$

for each $i = m+1, \ldots, m+l$, where π is the map from $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ to $\mathcal{A}_{\mathcal{N}_{\mathcal{F},\mathcal{M}}}$. We compute now the explicit form of the partial holomorphic connection induced by the universal connection. Indeed, let v belong to $\mathcal{F}$ and s belong to $\mathcal{N}_{\mathcal{F},M}$; since $\psi(v)$ belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$, if we take a lift $\tilde{s}$ of s to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$, i.e. $\mathrm{pr} \circ \Theta_1(\tilde{s}) = s$, we have that the partial holomorphic connection $(\delta, \mathcal{F})$ along $\mathcal{F}$ induced by the universal holomorphic connection for $\mathcal{N}_{\mathcal{F},M}$ is given by:

$$\delta_v(s) = \tilde{X}_{\psi(v)}(s) = \mathrm{pr}([\psi(v), \tilde{s}]).$$

We prove now this partial holomorphic connection is flat; indeed

$$\delta_u(\delta_v(s)) - \delta_v(\delta_u(s)) - \delta_{[u,v]}((s)) = \mathrm{pr}([\tilde{u}, [\tilde{v}, \tilde{s}]] - [\tilde{v}, [\tilde{u}, \tilde{s}]] - [[\tilde{u}, \tilde{v}], \tilde{s}]) = 0,$$

by the Jacobi identity. □

Remark 5.1.2. If $\mathcal{F}$ is a rank l foliation of the first infinitesimal neighborhood Corollary 4.1.4 tells us that there exists an atlas adapted to S and $\mathcal{F}|_S$ such that on each coordinate neighborhood such that $U_\alpha \cap S \neq \emptyset$ we have that $\mathcal{F}$ is generated by

$$[1]_2 \otimes \frac{\partial}{\partial z_\alpha^{m+1}}, \ldots, [1]_2 \otimes \frac{\partial}{\partial z_\alpha^{m+l}}.$$

In each coordinate patch, given a section $s = g^u \partial_{\alpha,u}$ of $\mathcal{N}_{\mathcal{F},M}$ we have that

$$\delta_{\partial/\partial z_\alpha^j}(g^u \partial_{\alpha,u}) = \frac{\partial g^u}{\partial z_\alpha^j} \partial_{\alpha,u}.$$

5.2 Action of subsheaves of $\mathcal{F}$ on $\mathcal{N}_{\mathcal{F},M}$

As usual let $\mathcal{F}$ be a foliation of S: in this section we shall discuss how the existence of coherent subsheaves of $\mathcal{T}_{S(1)}$ that restricted to S are subsheaves of $\mathcal{F}$ gives rise to variation actions on $\mathcal{N}_{F,M}$.

Lemma 5.2.1. *Let $\mathcal{E}$ be a coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$. Then $\mathcal{E}$ is a subsheaf of $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$.*

Proof. Let $\{U_\alpha, z_\alpha\}$ be an atlas adapted to S and $\mathcal{F}$. On each coordinate chart, a section v of $\mathcal{E}$ can be written as:

$$[a^u]_2 \frac{\partial}{\partial z^u} + [a^i]_2 \frac{\partial}{\partial z^i},$$

with $a^u \in \mathcal{I}_S$. Therefore, thanks to Remark 3.4.4, we know that v belongs to $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$. □

Definition 5.2.2. Let $\mathcal{E}$ be a coherent subsheaf of $\mathcal{T}_{S(1)}$; we say it is **S-faithful** if the restriction map $|_S : \mathcal{E} \to \mathcal{E}|_S$ is injective.

Proposition 5.2.3. *Suppose $\mathcal{E}$ is a coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$, S-faithful. Then there exists a partial holomorphic connection $(\delta, \mathcal{E})$ for $\mathcal{N}_{\mathcal{F},M}$.*

Proof. Since there are no generators sent to 0 by the restriction to S, then $\mathcal{E}|_{S\cap U_\alpha}$ is generated by $v_{k,\alpha} := \tilde{v}_{k,\alpha}|_S$. Please keep in mind that the generators of $\mathcal{E}|_S$ are always the restriction of the generators of $\mathcal{E}$, so, chosen the local generators of $\mathcal{E}$ we have a canonical way to extend the local generators of $\mathcal{E}|_S$.

Let π be the projection from $\mathcal{T}^{\mathcal{F}}_{M,S(1)}$ to $\mathcal{A}$ and w a section of $\mathcal{E}|_S$; we define a map $\tilde{\pi} : \mathcal{E}|_S \to \mathcal{A}$ by $\pi(w) := \pi(\tilde{w})$, where $\tilde{w}$ is an extension of w as a section of $\mathcal{E}$. On a trivializing neighborhood for $\mathcal{E}|_S$ a section has the following form:

$w = [f^k]_1 v_{k,\alpha} \in \mathcal{E}\,|_{S\cap U_\alpha}$. The difference between two representatives $\tilde{w}_1$ and $\tilde{w}_2$ of w in $\mathcal{E}$ on U_α can be written in the following form:

$$[g^k]_2 \tilde{v}_{k,\alpha}$$

where the g^k belong to $\mathcal{I}_S/\mathcal{I}_S^2$ and therefore this section belongs to $\mathcal{I}_S\,\mathcal{T}^{\mathcal{F}}_{M,S(1)}$. Therefore the map $\tilde{\pi}$ does not depend on the extension chosen.

Suppose now we have a section w of $\mathcal{E}|_S$ and two coordinate charts U_α and U_β on which the section is represented as $w_\alpha = [f^k_\alpha]_1 v_{k,\alpha}$ and $w_\beta = [f^k_\beta]_1 v_{k,\beta}$. Now, we have that, since $\mathcal{E}$ is a subbundle of $\mathcal{T}_{S(1)}$

$$\tilde{v}_{k,\alpha} = [(h_{\alpha\beta})^h_k]_2 \tilde{v}_{h,\beta},$$

which implies also that:

$$[f^k_\alpha (h_{\alpha\beta})^h_k]_1 = [f^h_\beta]_1.$$

We take two extensions $\tilde{w}_\alpha$ and $\tilde{w}_\beta$ on U_α and U_β respectively: we claim their difference lies in $\mathcal{I}_S\,\mathcal{T}^{\mathcal{F}}_{M,S(1)}$. We compute:

$$\begin{aligned}(\tilde{w}_\beta - \tilde{w}_\alpha)|_S &= ([\tilde{f}^k_\alpha]_2 \tilde{v}_{k,\alpha} - [\tilde{f}^h_\beta]_2 \tilde{v}_{h,\beta})|_S \\ &= ([\tilde{f}^k_\alpha]_2 [h^h_{\alpha\beta,k}]_2 \tilde{v}_{h,\beta} - [\tilde{f}^h_\beta]_2 \tilde{v}_{h,\beta})|_S \\ &= \big[[f^k_\alpha (h_{\alpha\beta})^h_k]_2 - [f^h_\beta]_2\big]_1 v_{h,\beta} = [0]_1.\end{aligned}$$

As stated, the difference between the two extensions lies in $\mathcal{I}_S\,\mathcal{T}^{\mathcal{F}}_{M,S(1)}$. So, the map $\tilde{\pi} : \mathcal{E}|_S \to \mathcal{A}$ is an $\mathcal{O}_S$-morphism between $\mathcal{E}\,|_S$ and $\mathcal{A}$ giving a splitting of the following sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_{\mathcal{F},M}) \longrightarrow \mathcal{A}_{\mathcal{F},\mathcal{E}\,|_S} \xrightarrow{\Theta_1} \mathcal{E}\,|_S \longrightarrow 0\,.$$

where $\mathcal{A}_{\mathcal{F},\mathcal{E}\,|_S}$ is the preimage of $\mathcal{E}\,|_S$ in $\mathcal{A}$ through Θ_1. □

Therefore, recalling Section 3.1 and Section 3.4 we have that there is a partial holomorphic connection on $\mathcal{N}_{F,M}$ along $\mathcal{E}\,|_S$, given as follows:

$$\delta_v(s) = \tilde{X}_{\pi(\tilde{v})}(s),$$

where $\tilde{X}$ is the universal connection on $\mathcal{A}_{\mathcal{N}_{\mathcal{F},M}}$.

Remark 5.2.4. This connection may not be flat. Therefore we can use the Bott Vanishing theorem only in its weak form. But, in case $\mathcal{E}$ is involutive a stronger result holds.

Corollary 5.2.5. *Suppose $\mathcal{E}$ is an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$ and S-faithful. Then there exists a flat partial holomorphic connection $(\delta, \mathcal{E})$ for $\mathcal{N}_{\mathcal{F},M}$.*

Proof. From Proposition 5.2.3 we already know there exists a partial holomorphic connection along $\mathcal{E}$; since $\mathcal{E}$ is involutive we can check if it is flat:

$$\delta_u(\delta_v(s)) - \delta_v(\delta_u(s)) - \delta_{[u,v]}((s)) = \mathrm{pr}([\tilde{u}, [\tilde{v}, \tilde{s}]] - [\tilde{v}, [\tilde{u}, \tilde{s}]] - [[\tilde{u}, \tilde{v}], \tilde{s}]) = 0,$$

by the Jacobi identity. □

Remark 5.2.6. In the paper [4] is defined the notion of Lie Algebroid morphism; given an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$ the splitting that gives rise to the partial holomorphic connection is a Lie algebroid morphism and last Corollary mirrors the fact that the universal partial holomorphic connection is flat (Proposition 3.4.12).

Corollary 5.2.7. *Suppose $\mathcal{E}$ is an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$, whose restriction to S is a foliation of S and is S-faithful. Then there exists a flat partial holomorphic connection $(\delta, \mathcal{E})$ for $\mathcal{N}_S$.*

Proof. If we take $\mathcal{F} = \mathcal{T}_S$ in Corollary 5.2.5 the assertion follows. □

5.3 The results of ABT for the normal bundle to a subvariety

In this section we give a survey of the results obtained by Abate, Bracci and Tovena in the series of paper [3], [4], [5] for the characteristic classes of the normal bundle to a submanifold S. The methods we have been using in this thesis are particularly connected to those in [4]; indeed, in that paper a more concrete version of the Atiyah sheaf for the normal bundle to a submanifold was constructed and many index theorems were deduced. This section is essentially an overview of the results in [4] with a couple of minor remarks.

Definition 5.3.1. Let M be a complex manidolf and let S be a complex submanifold. The **Atiyah sheaf of** S **in** M is the Atiyah sheaf for the foliation $\mathcal{T}_S$ of S (Definition 3.4.8).

We want now to understand what happens when the ambient manifold M admits a foliation $\mathcal{F}$. The main situations in which we can find us are mainly two: when the foliation leaves the submanifold S invariant, i.e., $\mathcal{F}|_S \subset \mathcal{T}_S$ and when $\mathcal{F}$ is transversal to S. When $\mathcal{F}$ leaves S invariant we are exactly in the case treated in section 5.2 Corollary 5.2.7 and therefore there exists a partial holomorphic connection for $\mathcal{N}_S$ along $\mathcal{F}$. The cases that interest us the most are the transversal cases; suppose S is splitting in M and we have a $\mathcal{F}$-faithful splitting morphism σ^*, then we can project the foliation $\mathcal{F}|_S \subset \mathcal{T}_{M,S}$ to a foliation $\mathcal{F}^\sigma := \sigma^*(\mathcal{F})$ of S, thanks to the splitting morphism $\sigma^* : \mathcal{T}_{M,S} \to \mathcal{T}_S$. We want to understand under which conditions this projection gives rise to a partial holomorphic connection on $\mathcal{N}_S$ along $\mathcal{F}$. Indeed, let T the morphism arising from the following composition:

$$T : \mathcal{F}^\sigma \xrightarrow{(\sigma^*|_{\mathcal{F}_S})^{-1}} \mathcal{F}_S \xrightarrow{\iota} \mathcal{T}_{M,S} \xrightarrow{\text{pr}} \mathcal{N}_S,$$

where pr is the canonical projection from $\mathcal{T}_{M,S}$ to $\mathcal{N}_S$ and with $\mathcal{F}_S$ we denote $\mathcal{F}|_S$. In the points x in S where $\mathcal{F}_x \subset \mathcal{T}_{S,x}$ we have that $T_x \equiv 0$ and this morphism is non zero if and only if the foliation $\mathcal{F}$ is transversal to S.

Remark 5.3.2. Please note that a great deal of work on the morphism T seen as a section of the bundle $\text{Hom}(F, N_S)$ and on index theorems arising from its existence (the Tangential index) was done in [21].

Now, under the hypothesis that S is split and comfortably embedded, in [4, Proposition 7.13] Abate, Bracci and Tovena investigate the existence and the properties of a class $\mathfrak{f}$ in $H^1(S, \mathcal{N}_S \otimes \mathcal{F}^\sigma \otimes (\mathcal{F}^\sigma)^*)$ defined in the following way. Given an atlas adapted to $\mathcal{F}^\sigma$ and S they find a special frame for $\mathcal{F}$ on each U_α given by $\{v_{\alpha,1}, \dots, v_{\alpha,l}\}$ such that $\sigma^*(v_{\alpha,i} \otimes [1]_1) = \partial/\partial z^i_\alpha$ in the form:

$$v_{\alpha,i} = \frac{\partial}{\partial z^i_\alpha} + (a_\alpha)^t_i \frac{\partial}{\partial z^t_\alpha}.$$

A generator $\tilde{v}_{\alpha,i}$ of $\mathcal{F}$ is written in such an atlas as

$$\tilde{v}_{\alpha,i} = (b_\alpha)^t_i \frac{\partial}{\partial z^t_\alpha} + (b_\alpha)^j_i \frac{\partial}{\partial z^j_\alpha};$$

the condition that $\sigma^*(\tilde{v}_{\alpha,i} \otimes [1]_1) = \partial/\partial z^i_\alpha$ forces $(b_\alpha)^{p'}_i$ to belong to $\mathcal{I}_S$, for $p' = m+l+1, \dots, n$, $i = m+1, \dots, m+l$. Moreover, we have that $[(b_\alpha)^j_i]_1$ is the identity matrix; hence $(b_\alpha)^j_i$ is an invertible matrix of germs. If we multiply the l-uple $\{v_{\alpha,m+1}, \dots, v_{\alpha,m+l}\}$ by its inverse we obtain a l-uple of elements of $\mathcal{F}$ of the desired form. Now, if we denote by $(c_{\beta\alpha})$ the cocycle defining the foliation $\mathcal{F}$, i.e. $v_{\beta,i} = (c_{\beta\alpha})^j_i v_{\beta,j}$ we have that

$$\begin{cases} (c_{\beta\alpha})^i_j = \frac{\partial z^i_\alpha}{\partial z^j_\beta} + (a_\beta)^t_i \frac{\partial z^i_\alpha}{\partial z^t_\beta}, \\ (c_{\beta\alpha})^i_j (a_\alpha)^u_i = \frac{\partial z^u_\alpha}{\partial z^j_\beta} + (a_\beta)^t_i \frac{\partial z^u_\alpha}{\partial z^t_\beta}. \end{cases} \tag{5.1}$$

An important remark that follows from these equalities is that the vector bundles associated to $\mathcal{F} \otimes \mathcal{O}_S$ and $\mathcal{F}^\sigma$ are represented by the same cocycle

$$[(c_{\beta\alpha})^i_j]_1 = \left[\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right]_1$$

in the frames $\{v_{\alpha,m+1}, \dots, v_{\alpha,m+l}\}$ and $\{\partial/\partial z^{m+1}_\alpha, \dots, \partial/\partial z^{m+l}_\alpha\}$ respectively. The class $\mathfrak{f}$ is represented by the cocycle:

$$f_{\beta\alpha} = \left[(c_{\alpha\beta})^i_j\right]_1 \tilde{\rho}\left(\left[(c_{\beta\alpha})^{\tilde{i}}_i\right]_2\right) \otimes \frac{\partial}{\partial z^{\tilde{i}}} \otimes \omega^j_\alpha.$$

Remark 5.3.3. This class seems to be the obstruction to the splitting of the sequence:

$$0 \longrightarrow \mathcal{I}_S / {\mathcal{I}_S}^2 \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{O}_{S(1)} \otimes_{\mathcal{O}_S} \mathcal{F} \xrightarrow{\theta_1 \otimes \sigma^*} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{F}^\sigma \longrightarrow 0\,,$$

where $\mathcal{F}^\sigma \cong \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{F}^\sigma$. In some way, this could be an obstruction to the extendability of $\mathcal{F}^\sigma$ to the first infinitesimal neighborhood. This remark only occured to our attention during the writing of this thesis and is worth further investigation.

If T^* is the morphism induced in cohomology by $\mathrm{id} \otimes T \otimes \mathrm{id}$ and S is comfortably embedded this class has image $T^*(\mathfrak{f})$ exactly the Atiyah class of the sequence:

$$0 \to \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S) \to \mathcal{A}_{\mathcal{N}_S} \to \mathcal{F}^\sigma \to 0.$$

This gives rise to different ways to approach the problem; indeed, the vanishing of the class $\mathfrak{f}$ implies the existence of a partial holomorphic connection.

Definition 5.3.4. Let S be a complex m-codimensional submanifold of an n-dimensional complex manifold M and let $\mathcal{F}$ be a holomorphic foliation $\mathcal{F}$ on M, of dimension $d \leq \dim S$. Assume that S splits into M with first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated projection $\sigma^* : \mathcal{T}_{M,S} \to \mathcal{T}_S$. We shall say that $\mathcal{F}$ **splits along** ρ if $\mathfrak{f} = 0$ in $H^1(S, \mathrm{Hom}(\mathcal{F}^\sigma, \mathcal{N}_S^* \otimes \mathcal{F}))$.

Now, since we are using an atlas adapted to ρ the first line of (5.1) yields:

$$\tilde{\rho}([(c_{\beta\alpha})_i^j]_2) = [(a_\beta)_i^s]_1 \left[\frac{\partial z_\alpha^j}{\partial z_\beta^s}\right]_2 + \left[\frac{\partial z^i}{\partial z^{\tilde{t}}}\right]_1 [(a_\beta)_i^{\tilde{t}}]_2,$$

where, exceptionally, $\tilde{t} = m+l+1, \ldots, n$. This permits to find a list of sufficient conditions under which the foliation $\mathcal{F}$ splits along ρ, allowing to prove the following.

Theorem 5.3.5 ([4] Theorem 7.21)**.** *Let S be a compact, complex, reduced, irreducible, possibly singular, subvariety of dimension d of an n-dimensional complex manifold M, and assume that S has extendable normal bundle. Let $\mathcal{F}$ be a (possibly singular) holomorphic foliation $\mathcal{F}$ on M, of dimension $l \leq d$. Assume there exists an analytic subset Σ of S containing $(S(\mathcal{F}) \cap S) \cup S^{sing}$ such that, setting $S^0 = S \setminus \Sigma$, we have either*

1. *$\mathcal{F}$ is tangent to S^0 and $\mathcal{F}|_{S^0}$ is a non singular holomorphic foliation of S^0; or*
2. *S^0 is comfortably embedded in M with respect to a first order lifting ρ which is $\mathcal{F}$-faithful outside of Σ, and*
   - *(a) S^0 is 2-linearizable and $l = \dim S$, or*
   - *(b) S^0 is 2-linearizable and there exists a nonsingular holomorphic foliation of S^0 transversal to $\mathcal{F}^\sigma$, or*
   - *(c) $\mathcal{F}|_{S^0(1)}$ is isomorphic to the trivial sheaf $\mathcal{O}^l_{S^0(1)}$ of dimension l or, more generally,*
   - *(d) $T^*(\mathfrak{f}) = 0$ in $H^1(S^0, \mathcal{N}^*_{S^0} \otimes \mathcal{N}_{S^0} \otimes (\mathcal{F}^\sigma)^*)$.*

Then, there exists a flat partial holomorphic connection for $\mathcal{N}_S$ along $\mathcal{F}$ on S^0 in case (1) or simply a partial holomorphic connection for $\mathcal{N}_S$ along $\mathcal{F}$ on S^0 in case (2).

5.4 Index theorems for foliations

Following the work [31] and the articles [3], [4], we know that the existence of a partial holomorphic connection, thanks to Bott's Vanishing Theorem 2.1.7, gives rise to the vanishing of some of the Chern classes of a vector bundle and therefore to an index theorem. In Section 3.4 we found a concrete realization of the Atiyah sheaf for the normal bundle of a foliation as a quotient of the ambient tangent bundle while in Section 5.1 we proved that the Atiyah sequence splits if there exists a foliation of the first infinitesimal neighborhood. In this section we state the index theorems that follow directly from our treatment.

The simpler case is when we have a foliation of the first infinitesimal neighborhood; then we have a partial holomorphic connection on $\mathcal{N}_{F,M}$ (Theorem

5.1.1) and so, Bott's Vanishing Theorem (Theorem 2.1.7) and Čech-de Rham theory permit us to prove the following.

Theorem 5.4.1. *Let S be a codimension m compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a rank l foliation on S, such that it extends to the first infinitesimal neighborhood of $S \backslash S(\mathcal{F})$, and let $S(\mathcal{F}) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $S(\mathcal{F})$ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n-m-l$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F},M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{F,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

Proof. If we denote by F the vector bundle associated to $\mathcal{F}$ we have that the virtual bundle associated with the sheaf $\mathcal{N}_{\mathcal{F},M}$ is nothing else but $[TM|_S - F|_S]$. Now, outside the singularity set of $\mathcal{F}$, this virtual bundle is a vector bundle on S and by Theorem 5.1.1 it admits a partial holomorphic connection along $\mathcal{F}|_S$; Bott Vanishing Theorem tells us that for each ϕ of degree k larger than $n-m-l$ we have that the restriction of $\phi(\mathcal{N}_{\mathcal{F},M})$ to $S \backslash S(\mathcal{F})$ is represented by the 0 form. Applying the localization process as in Section 2.2 the result follows. □

Using now the results of Section 5.2 we can prove a stronger result.

Theorem 5.4.2. *Let S be a codimension m compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a foliation on S and let $\mathcal{E}$ be a rank l subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S, is a subsheaf of $\mathcal{F}$. Suppose moreover that it is S-faithful. Let $\Sigma = S(\mathcal{F}) \cup S(\mathcal{E})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n-m-l+\lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{E}, \mathcal{N}_{F,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{F,M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{E}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{F,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

Remark 5.4.3. Please note that by Corollary 5.2.5 if $\mathcal{E}$ is involutive the above holds with $n-m-l$ instead of $n-m-l+\lfloor l/2 \rfloor$.

Suppose now we have a foliation of M, transversal to S, 2-splitting submanifold of M, and suppose we have a first order $\mathcal{F}$-faithful splitting σ_2 outside an algebraic subset. Now, the foliation $\mathcal{F}^{\sigma_2}$ is a foliation of the first infinitesimal neighborhood of S and by Theorem 5.4.1 we have the following.

Theorem 5.4.4. *Let S be a codimension m 2-splitting compact submanifold of a n dimensional complex manifold M. Let $\mathcal{F}$ be a rank l holomorphic foliation defined on a neighborhood of S. Suppose there is a 2-splitting first order $\mathcal{F}$-faithful outside an analytic subset Σ of U containing $S(\mathcal{F}) \cap S$ and that S is not contained in Σ. Let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k bigger* *than $n-m-l$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F}^\sigma,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F}^\sigma,M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F}^\sigma,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F}^\sigma,M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}^\sigma,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}^\sigma,M}$.

Remark 5.4.5. An interesting research path is to investigate the relation between $\mathcal{N}_{\mathcal{F}^\sigma,M}$ and $\mathcal{N}_{\mathcal{F}}|_S$. The motivation behind this question is easily seen: suppose M is a complex surface and $\mathcal{F}$ is a dimension 1 singular foliation transversal to S, a 2-splitting 1 dimensional submanifold. Suppose moreover that the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T}_M \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0$$

splits when restricted to S. Suppose we have a first order $\mathcal{F}$-faithful splitting σ^* outside Σ; thanks to the splitting of S and the splitting of the sequence above σ^* induces an isomorphism between $\mathcal{N}_{\mathcal{F}}|_S$ and $\mathcal{N}_{\mathcal{F}^\sigma,M}$ outside the singular points of $\mathcal{F}$. Suppose $\mathcal{F}$ admits an algebraic compact leaf L. If we denote by $\mathcal{N}_L$ the normal sheaf to this leaf we have that $\mathcal{N}_L \equiv \mathcal{N}_{\mathcal{F}}|_L$ and we have that

$$\int_S c_1(N_{\mathcal{F}^\sigma,M}) = \int_S c_1(N_{\mathcal{F}}|_S) = \int_S c_1(N_L) = (L \cdot S)$$

is the intersection number between L and S. Therefore we could apply this test to foliations, getting informations on the intersection numbers of possible analytic leaves.

The other results follow from the splitting of the sequence (4.6) studied in Section 4.4. In case $\mathcal{F}$ has rank 1 and we do not need to take care of involutivity we have the following consequence of 5.4.1.

Theorem 5.4.6. *Let S be a codimension m compact submanifold splitting in an n dimensional complex manifold M, and suppose $\mathcal{F}$ is a rank 1 holomorphic foliation defined on S. Suppose sequence (4.6) splits and let $\Sigma = S(\mathcal{F})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree $n-m$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) \in H_0(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F},M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{\mathcal{F},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

Now, if $\mathcal{F}$ has rank l and we suppose its extension arising from the splitting of (4.6) is involutive we have the following.

Theorem 5.4.7. *Let S be a codimension m compact submanifold splitting in M, n dimensional complex manifold, and suppose $\mathcal{F}$ is a rank l holomorphic foliation defined on S. Suppose sequence (4.6) splits and that the image of $\tilde{\mathcal{F}}/\mathcal{V}$ in $\tilde{\mathcal{F}}$ is involutive. Let $\Sigma = S(\mathcal{F})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ*

of degree k larger than $n-m-l$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{F,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{F,M}$ near Σ_λ such that:

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{\mathcal{F},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

In case we drop the involutivity assumption we have a weaker form thanks to Theorem 5.2.3.

Theorem 5.4.8. *Let S be a codimension m compact submanifold splitting in M complex manifold of dimension n. Suppose $\mathcal{F}$ is a foliation of S of rank l and suppose sequence (4.6) splits. Let $\Sigma = S(\mathcal{F})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of its singular set in connected components. Then, for every symmetric homogeneous polynomial ϕ of degree k larger than $n-m-l+\lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\alpha) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{\mathcal{F},M}$ near Σ_α such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{\mathcal{F},M}; \Sigma_\alpha) = \int_S \phi(\mathcal{N}_{\mathcal{F},M}),$$

where $\phi(\mathcal{N}_{\mathcal{F},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F},M}$.

In the case S has first order extendable tangent bundle the vanishing of the cohomology class associated to (4.6) follows directly from Corollary 4.4.5, but we cannot say anything about the involutivity of this extension.

Theorem 5.4.9. *Let S be a codimension m compact submanifold splitting in an n dimensional complex manifold M, and with first order extendable tangent bundle. Let $\mathcal{F}$ be a rank l holomorphic foliation defined on S. Let $\Sigma = S(\mathcal{F})$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n-m-l+\lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{F}, \mathcal{N}_{F,M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}$ and $\mathcal{N}_{F,M}$ near Σ_λ such that:*

$$\sum_\lambda Res_\phi(\mathcal{F}, \mathcal{N}_{F,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{F,M}),$$

where $\phi(\mathcal{N}_{F,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{F,M}$.

Remark 5.4.10. From the theory developed in Section 4.4 it seems likely that, given a foliation $\mathcal{F}$ of the first infinitesimal neighborhood and an involutive subsheaf $\mathcal{G}$ of rank l of $\mathcal{F}|_S$ this subsheaf extends to a subsheaf of $\mathcal{F}$, possibly non involutive. This does not give rise to new index theorems, but is indeed worth noting and investigating.

5.5 Computing the variation index: the tangential case

In this section we will compute the residue for a codimension 1 foliation of the first infinitesimal neighborhood of a codimension 1 submanifold in a surface.

Let (U_1, x, y) be a neighborhood of 0 in $\mathbb{C}^2$, let $S = \{x = 0\}$; let $\mathcal{F}$ be a foliation of $S(1)$ such that $\operatorname{Sing}(\mathcal{F}) = \{0\}$ and let v be a generator of $\mathcal{F}$; that is a holomorphic section of $\mathcal{T}_{S(1)}$ with an isolated singularity in 0. Supposing $\mathcal{F}$ reduced, from Section 4.2 and Remark 4.2.5 we see that this is assumption does not give rise to a loss of generality for our computation.

Remark 5.5.1. Please note also that, if we denote by $\tilde{v}$ an extension of v to U_1 and by $\tilde{\mathcal{F}}$ the foliation generated by it, thanks to how we defined the holomorphic action and the theory developed for local extensions, the computation of this residue could be reduced to the computation of the residue given by the Lehmann-Khanedani-Suwa action of $\tilde{v}$ on $\mathcal{N}_{\mathcal{F}}|_S$, which can be found e.g. in [31, Ch. IV, Theorem 5.3].

We will, anyway, compute the index explictly in the framework we developed. Call $U_0 := U_1 \setminus \{0\}$; with an abuse of notation we will also say $M := U_1$. Let G be the trivial line bundle on S; we can see $v|_S$ as a holomorphic homomorphism between G and TS. On U_0 we can see G as a subbundle of $TM|_S$, moreover G embedded through $v|_S$ is nothing else that the bundle associated to $\mathcal{F}|_S$. Therefore, we can speak of the virtual bundle $[TM|_S - G]$, which coincides, on U_0, with the normal bundle to the foliation $\mathcal{F}|_{U_0 \cap S}$ in the ambient tangent bundle $TM|_{U_0 \cap S}$, denoted by $N_{\mathcal{F},M}$. Since the only homogeneous symmetric polynomial in dimension 1 is the trace we would like to compute the residue for the first Chern class of $[TM|_S - G]$, whose sheaf of sections is $\mathcal{N}_{\mathcal{F},M}$. Being the first Chern class additive, we are going to compute $c_1(TM|_S) - c_1(G)$. If U_0 is small enough, thanks to the embedding of G into $TM|_S$ we have that on U_0 we can see $TM|_S$ as the direct sum $G \oplus N_{\mathcal{F},M}$. We are going to apply Proposition 2.1.13 to the following sequence:

$$0 \longrightarrow F|_{U_0 \cap S} \longrightarrow TM|_{U_0 \cap S} \longrightarrow N_{\mathcal{F},M} \longrightarrow 0.$$

We want to build on U_0 a family of connections compatible with the sequence, so that Theorem 2.1.14 implies that $c_1(N_{\mathcal{F},M})$ on U_0 is 0. We proved that the existence of a foliation of the first infinitesimal neighborhood gives rise to a partial connection on $N_{\mathcal{F},M}$ along $\mathcal{F}$. Now, thanks to Theorem 5.1.1 we can compute the actual connection matrix of this partial holomorphic connection on $\mathcal{N}_{\mathcal{F},M}$ and extend it to a connection on $N_{\mathcal{F},M}$, denoted by ∇. To build a family of connections simplifying our computations we take on $U_0 \cap S$ the connection ∇_0^G which is trivial with respect with the generator 1_G of the trivial line bundle G. Since $TM|_S$ on $U_0 \cap S$ is the direct sum of G and $N_{\mathcal{F},M}$ we let the connection for $TM|_S$ be the direct sum connection $\nabla_0^{TM} := \nabla \oplus \nabla_0^G$. Both ∇_0^{TM} and ∇_0^G are holomorphic connections along F, therefore we can apply Bott's Vanishing in the version for virtual bundles and obtain that $c_1(N_{\mathcal{F},M}) \equiv 0$ on U_0.

In Čech-de Rham cohomology relative to the cover $\{U_0, U_1\}$ the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ is represented as a triple $(\omega_0, \omega_1, \sigma_{01})$, where ω_0 is the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ on U_0, ω_1 is the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ on U_1 while σ_{01} is a 1-form, the Bott difference form, i.e., a 1-form such that $\omega_1 - \omega_0 = d\sigma_{01}$ on $U_0 \cap U_1$(for a complete treatment, refer to Lemma 1.6.12). Due to the additivity of the first Chern class, to compute the first Chern class of $\mathcal{N}_{F,M}$ we need to compute the first Chern classes of G and $TM|_S$ on U_1 (we already know the first Chern class of $\mathcal{N}_{F,M}$ on U_0 is 0) and the Bott difference forms $c_1(\nabla_0^{TM}, \nabla_1^{TM})$ and $c_1(\nabla_0^G, \nabla_1^G)$. On U_1 we can take, again, as a connection for G the connection which is trivial with respect to the generator 1_G of G: therefore $c_1(\nabla_0^G, \nabla_1^G) = 0$, since the connections for G on U_0 and U_1 are the same. On U_1 we take as ∇_1^{TM} the $\partial/\partial x$, $\partial/\partial y$ trivial connection; then $c_1(\nabla_1^{TM}) = 0$ and the problem reduces to compute the Bott difference form $c_1(\nabla_0^{TM}, \nabla_1^{TM})$. To compute it we need the connection matrix for ∇_0^{TM} with respect to the frame $\partial/\partial x$, $\partial/\partial y$. First of all we compute the action of ∇ on the equivalence class $\nu = [\partial/\partial x]$ in $N_{\mathcal{F},M}$. The generator v of $\mathcal{F}$ is written in coordinates as

$$[A]_2 \frac{\partial}{\partial x} + [B]_2 \frac{\partial}{\partial y},$$

where $[A]_2$ belongs to $\mathcal{I}_S/\mathcal{I}_S^2$. In the following, we shall denote by v_S the restriction of $\tilde{v}$ to S; in coordinates we have that $v_S = [B]_1 \partial/\partial y$. We compute now the action of F on $\mathcal{N}_{\mathcal{F},M}$, recalling Theorem 5.1.1

$$\nabla_{v_S}(\nu) = \mathrm{pr}\left(\left[[A]_2 \frac{\partial}{\partial x} + [B]_2 \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]\Big|_S\right) = -\left[\frac{\partial A}{\partial x}\right]_1 \nu.$$

We compute now the connection matrix for ∇. Since

$$-\left[\frac{\partial A}{\partial x}\right]_1 = ([C]_1 \cdot dx + [D]_1 \cdot dy)([B]_1 \frac{\partial}{\partial y}) = [D \cdot B]_1,$$

it follows that the connection matrix is nothing else but:

$$\omega = -\left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy.$$

We have now all the tools needed to compute the connection matrix for ∇_0^{TM}:

$$\begin{aligned} \nabla_0^{TM}\left(\frac{\partial}{\partial x}\right) &= \nabla(\nu) = -\left[\frac{\partial A}{\partial x} \cdot \frac{1}{B}\right]_1 dy \otimes \frac{\partial}{\partial x}, \\ \nabla_0^{TM}\left(\frac{\partial}{\partial y}\right) &= \nabla_0^G\left(\frac{1}{B} \cdot v\right) = -\left[\frac{dB}{B^2}\right]_1 \cdot v = -\left[\frac{dB}{B}\right]_1 \otimes \frac{\partial}{\partial y}. \end{aligned}$$

Thus the connection matrix has the following form:

$$\begin{bmatrix} -\left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy & 0 \\ 0 & -\left[\frac{dB}{B}\right]_1 \end{bmatrix}.$$

We can compute now the Bott difference form. We consider the bundle $TM \times [0,1] \to M \times [0,1]$ and the connection $\tilde{\nabla}$ given by $\tilde{\nabla} := (1-t)\nabla_0^{TM} + t\nabla_1^{TM}$. The connection matrix for $\tilde{\nabla}$ is given by:

$$(1-t) \cdot \begin{bmatrix} -\left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy & 0 \\ 0 & -\left[\frac{dB}{B}\right]_1 \end{bmatrix}.$$

The curvature matrix for $\tilde{\nabla}$ is:

$$\begin{bmatrix} dt \wedge \left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy & 0 \\ 0 & dt \wedge \left[\frac{dB}{B}\right]_1 \end{bmatrix}.$$

The Bott difference form is given by $\pi_*(c_1(\tilde{\nabla}))$ where π_* is integration along the fibre of the projection $\pi : M \times [0,1] \to M$. The Bott difference form is then:

$$\left[\frac{1}{B}\frac{\partial A}{\partial x}\right]_1 dy + \left[\frac{dB}{B}\right]_1.$$

So, the residue for $c_1(\mathcal{N}_{\mathcal{F},M})$ in 0 is

$$\frac{1}{2\pi\sqrt{-1}}\int_{\{x=0,|y|=\varepsilon\}}\left[\frac{1}{B}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)\right]_1 dy.$$

Remark 5.5.2. Already with slightly harder examples the computations of indices turn out to be really complicated; please note that while Remark 5.5.1 tells us that in higher dimension the computation follows almost directly from known results it would be interesting to compute the indices when dealing with singularities that are not isolated.

5.6 Computing the variation index: the transverse case

Let (U_1, x, y) be a neighborhood of 0 in $\mathbb{C}^2$, let $S = \{x = 0\}$. Let now v be a holomorphic section of $\mathcal{T}_{M,S(1)}$ with an isolated singularity in 0. As before, we call $U_0 := U_1 \setminus \{0\}$ and $M = U_1$. Please remark that we drop the hypothesis about v belonging to $\mathcal{T}_{S(1)}$. We want to compute the variation index for such a foliation. Since the situation is local we can assume we have a local 2 splitting, first order $\mathcal{F}$-faithful outside 0 and that we are in a chart adapted to it and therefore we have a map $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$. Write $\tilde{v}$ in coordinates as:

$$\tilde{v} = [A(x,y)]_2\frac{\partial}{\partial x} + [B(x,y)]_2\frac{\partial}{\partial y}.$$

Now we can write $[A(x,y)]_2 = [\tilde{\rho}([A(x,y)]_2) + R(x,y)]_2$, where $\tilde{\rho}$ is the θ_1 derivation associated to the 1-splitting induced by the 2-splitting; then,

$$\sigma^*(\tilde{v}) = (\tilde{\rho}([A(x,y)]_2)\partial/\partial x + B(x,y)\partial/\partial y.$$

Moreover, we have a splitting $\sigma^* : \mathcal{T}_{M,S} \to \mathcal{T}_S$, givings rise on $U_0 \cap S$ to an isomorphism between $\mathcal{F}_S$, the sheaf of germs of sections of the foliation generated by $v_S := v|_S$ and the sheaf of germs of sections of $\mathcal{F}^\sigma$. Now, the vector field

$$w = [\tilde{\rho}([A(x,y)]_2)]_2\partial/\partial x + [B(x,y)]_2\partial/\partial y$$

is a section of $\mathcal{T}_{S(1)}$, giving rise to a foliation of the first infinitesimal neighborhood. We can now compute the index as in the former section: the residue for $c_1(N_{\mathcal{F}^\sigma,M})$ is therefore:

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\int_{\{|y|=\varepsilon\}}&\left[\frac{1}{B}\left(\frac{\partial[\tilde{\rho}([A]_2)]_2}{\partial x}+\frac{\partial B}{\partial y}\right)\right]_1 dy\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\{|y|=\varepsilon\}}\left[\frac{1}{B}\left(\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\cdot x\right)+\frac{\partial B}{\partial y}\right)\right]_1 dy\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\{|y|=\varepsilon\}}\left[\frac{1}{B}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)\right]_1 dy.\end{aligned}$$

Remark 5.6.1. The term

$$\frac{\partial^2 A}{\partial x^2} \cdot x$$

in the last computation disappears since it belongs to $\mathcal{I}_S$.

5.7 Action of a subsheaf on the normal bundle on its involutive closure

In this section, given a coherent subsheaf $\mathcal{E}$ of $\mathcal{T}_{S(1)}$ we shall define a natural object, its involutive closure, the smallest involutive subsheaf containing $\mathcal{E}$. Thanks to the machinery developed in the former sections, in particular in Section 5.2, it is proved that the existence of $\mathcal{E}$ gives rise to vanishing theorems for its involutive closure.

Definition 5.7.1. Let $\mathcal{E}$ be a coherent subsheaf of $\mathcal{T}_{S(1)}$ such that $\mathcal{E}\,|_S$ is non empty. We denote by $\mathrm{Sing}(\mathcal{E})$ the set $\{x \in S \mid \mathcal{T}_{S(1)} \,/\, \mathcal{E} \text{ is not } \mathcal{O}_{S(1),x} \text{ −free}\}$. On $S \setminus \mathrm{Sing}(\mathcal{E})$ we define the **involutive closure** $\mathcal{G}$ of $\mathcal{E}$ in S to be the intersection of all the coherent involutive subsheaves of $\mathcal{T}_S$ containing $\mathcal{E}\,|_S$.

Recall that the intersection of coherent subsheaves of $\mathcal{T}_S$ is again a coherent subsheaf of $\mathcal{T}_S$; now, $\mathcal{G}$ is involutive by definition and therefore gives rise to a foliation of S. Clearly, $\mathcal{E}\,|_S$ is a subsheaf of $\mathcal{G}$ and we can apply Proposition 5.2.3 getting the following result.

Theorem 5.7.2. *Let S be a codimension m compact submanifold of M complex manifold of dimension n. Suppose $\mathcal{E}$ is a coherent subsheaf of $\mathcal{T}_{S(1)}$ of rank l, whose restriction $\mathcal{E}\,|_S$ has rank l. Let $\mathcal{G}$ be the involutive closure of $\mathcal{E}$ in S. Let $\Sigma = S(\mathcal{E}) \cup S(\mathcal{G}) = \bigcup_\alpha \Sigma_\alpha$ be the decomposition of Σ in connected components. Then, for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{E}\,|_S, \mathcal{N}_{\mathcal{G},M}; \Sigma_\alpha) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{E}\,|_S$ and $\mathcal{N}_{\mathcal{G},M}$ near Σ_α such that:*

$$\sum_\lambda Res_\phi(\mathcal{E}\,|_S, \mathcal{N}_{\mathcal{G},M}; \Sigma_\alpha) = \int_S \phi(\mathcal{N}_{\mathcal{G},M}),$$

where $\phi(\mathcal{N}_{\mathcal{G},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{G},M}$.

Chapter 6

Holomorphic self-maps

Remark 6.0.3. In this chapter we follow the Einstein summation convention; for an explanation of the different ranges of the indices, refer to Section 1.1.

6.1 The canonical section

We recall in this section some of the results of [3]. Suppose we have a complex manifold M of complex dimension n and a holomorphic mapping f fixing pointwise a complex submanifold S. In this section we denote by $\mathrm{Sym}^k(\mathcal{N}_S) := \mathcal{N}_S^{\otimes k}$ the k-th tensor power of $\mathcal{N}_S$.

Definition 6.1.1. Let $f \in \mathrm{End}(M, S)$, $f \neq \mathrm{id}_M$. The **order of contact** ν_f of f with S is defined by

$$\nu_f = \min_{h \in \mathcal{O}_{M,p}} \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}^{\mu}_{S,p}\} \in \mathbb{N}^*$$

where p is any point of S.

In [3], it is proved that this number is well defined and that it can be computed using the fact that

$$\nu_f = \min_{j=k,\dots,n} \max\{\mu \in \mathbb{N} \mid f^k_\alpha - z^k_\alpha \in \mathcal{I}^{\mu}_{S,p}\},$$

where (U_α, z_α) is a local chart in p and $f^k_\alpha = z^k_\alpha \circ f$.

Remark 6.1.2. A straightforward computation tells us that

$$f^k_\alpha - z^k_\alpha = \frac{\partial z^k_\alpha}{\partial z^h_\beta}(f^h_\beta - z^h_\beta) + \frac{\partial^2 z^k_\alpha}{\partial z^{h_1}_\beta \partial z^{h_2}_\beta}(f^{h_1}_\beta - z^{h_1}_\beta)(f^{h_2}_\beta - z^{h_2}_\beta) + \dots;$$

so

$$f^k_\alpha - z^k_\alpha = \frac{\partial z^k_\alpha}{\partial z^h_\beta}(f^h_\beta - z^h_\beta) + \mathcal{I}^{2\nu_f}_S.$$

In every local chart we can define a local section of $\mathcal{I}^{\nu_f}_S/\mathcal{I}^{\nu_f+1}_S \otimes \mathcal{T}_{M,S}$ by:

$$X_{f,\alpha} = [f^j_\alpha - z^j_\alpha]_{\nu_f+1} \otimes \frac{\partial}{\partial z^j_\alpha}.$$

We shall prove below that this section is well defined globally on S.

Definition 6.1.3. Let $f \in \mathrm{End}(M,S)$, $f \neq \mathrm{id}_M$. The section of $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1} \otimes \mathcal{T}_{M,S}$ given in local coordinates as

$$X_{f,\alpha} = [f_\alpha^h - z_\alpha^h]_{\nu_f+1} \otimes \frac{\partial}{\partial z_\alpha^h},$$

for any local chart (U_α, z_α) at a point $p \in S$ is called the **canonical section** associated to f.

Definition 6.1.4. Let $f \in \mathrm{End}(M,S)$, $f \neq \mathrm{id}_M$. The **canonical distribution** $\mathcal{F}_f$ associated to f is the subsheaf of $\mathcal{T}_{M,S}$ defined by

$$\mathcal{F}_f = X_f(\mathrm{Sym}^{\nu_f}(\mathcal{N}_S)).$$

We shall say that f is **tangential** if $\mathcal{F}_f \subset \mathcal{T}_S$.

Remark 6.1.5. Since $f_\alpha^k - z_\alpha^k$ is in $\mathcal{I}_S^{\nu_f}$ we can find holomorphic functions $g^h_{\alpha,r_1,\dots,r_{\nu_f}}$ symmetric in the lower indices such that

$$f_\alpha^h - z_\alpha^h = g^h_{\alpha,r_1,\dots,r_{\nu_f}} z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}}.$$

These elements are not uniquely defined as elements of $\mathcal{O}_M$, since, if we find elements $e^j_{r_1,\dots,r_{\nu_f}}$ such that $(e_\alpha)^k_{r_1,\dots,r_{\nu_f}} z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}} = 0$, then

$$((g_\alpha)^k_{r_1\dots r_{\nu_f}} + (e_\alpha)^k_{r_1\dots r_{\nu_f}}) z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}} = (g_\alpha)^k_{r_1\dots r_{\nu_f}} z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}}.$$

Now, $(e_\alpha)^k_{r_1,\dots,r_{\nu_f}} z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}} = 0$ implies that $(e_\alpha)^k_{r_1,\dots,r_{\nu_f}}$ belongs to $\mathcal{I}_S$ and therefore the $(g_\alpha)^k_{r_1\dots r_{\nu_f}}$ are uniquely defined in $\mathcal{O}_S$.

Since S is a submanifold, for each pair of coordinate charts U_α and U_β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$ there exists $(h_{\alpha\beta})^r_s$ such that

$$z_\alpha^r = (h_{\alpha\beta})^r_s z_\beta^s.$$

Now, thanks to Remark 6.1.2 we have the following:

$$\begin{aligned} g^h_{\alpha,r_1,\dots,r_{\nu_f}} z_\alpha^{r_1} \cdots z_\alpha^{r_{\nu_f}} &= g^h_{\alpha,r_1,\dots,r_{\nu_f}} (h_{\alpha\beta})^{r_1}_{s_1} \cdots (h_{\alpha\beta})^{r_{\nu_f}}_{s_{\nu_f}} z_\beta^{s_1} \cdots z_\beta^{s_{\nu_f}} \\ &= \frac{\partial z_\alpha^h}{\partial z_\beta^k} g^k_{\beta,s_1,\dots,s_{\nu_f}} z_\beta^{s_1} \cdots z_\beta^{s_{\nu_f}} + \mathcal{I}_S^{2\nu_f}. \end{aligned}$$

Remark 6.1.6. If S is a codimension m submanifold we have that $\mathcal{F}_f$ is tangential if and only if in each coordinate patch $g^r_{\alpha,r_1,\dots,r_{\nu_f}} \in \mathcal{I}_S$ for each $r, r_1, \dots, r_{\nu_f} = 1, \dots, m$.

Remark 6.1.7. From now on we assume that

$$l := \mathrm{rk}_{\mathcal{O}_S} \mathrm{Sym}^{\nu_f}(\mathcal{N}_S) = \binom{m + \nu_f - 1}{\nu_f} \leq \dim S,$$

where m is the codimension of S.

Lemma 6.1.8. *Let $f \in End(M,S)$, $f \neq id_M$. The canonical distribution $\mathcal{F}_f \subset \mathcal{T}_{M,S}$ is well defined.*

Proof. The sheaf $\mathcal{F}_f$ is generated locally by the elements

$$v_{r_1\dots r_{\nu_f},\alpha} = [(g_\alpha)^k_{r_1\dots r_{\nu_f}}]_1 \frac{\partial}{\partial z^k_\alpha}.$$

We prove that the canonical section is well defined; we want to prove that $X_{f,\alpha} = X_{f,\beta}$. Indeed:

$$\begin{aligned} X_{f,\alpha} &= [(g_\alpha)^k_{r_1\dots r_{\nu_f}}]_1 dz^{r_1}_\alpha \otimes \cdots \otimes dz^{r_{\nu_f}}_\alpha \otimes \frac{\partial}{\partial z^k_\alpha} \\ &= [(g_\alpha)^k_{r_1\dots r_{\nu_f}}]_1 \frac{\partial z^{r_1}_\alpha}{\partial z^{s_1}_\beta} \cdots \frac{\partial z^{r_{\nu_f}}_\alpha}{\partial z^{s_{\nu_f}}_\beta} \frac{\partial z^h_\beta}{\partial z^k_\alpha} dz^{s_1}_\beta \otimes \cdots \otimes dz^{s_{\nu_f}}_\beta \otimes \frac{\partial}{\partial z^h_\beta}. \end{aligned}$$

Now, we have that $z^s_\beta = (h_{\alpha\beta})^s_r z^r_\alpha$; this implies that:

$$\begin{aligned} (h_{\alpha\beta})^{s_1}_{r_1} \cdots (h_{\alpha\beta})^{s_{\nu_f}}_{r_{\nu_f}} (g_\beta)^k_{s_1\dots s_{\nu_f}} z^{r_1}_\alpha \cdots z^{r_{\nu_f}}_\alpha &= (g_\beta)^k_{r_1\dots r_{\nu_f}} z^{r_1}_\beta \cdots z^{r_{\nu_f}}_\beta \\ &= z^k_\beta \circ f - z^k_\beta = (f^h_\alpha - z^h_\alpha) \frac{\partial z^k_\beta}{\partial z^h_\alpha} + R_{2\nu_f} \\ &= \frac{\partial z^k_\beta}{\partial z^h_\alpha} (g_\alpha)^h_{r_1\dots r_{\nu_f}} z^{r_1}_\alpha \cdots z^{r_{\nu_f}}_\alpha + R_{2\nu_f}, \end{aligned}$$

where $R_{2\nu_f}$ are elements of $\mathcal{I}^{2\nu_f}_S$. Now, we have that

$$[(h_{\alpha\beta})^s_r]_1 = \left[\frac{\partial z^r_\alpha}{\partial z^s_\beta}\right]_1.$$

This proves the assertion. □

Definition 6.1.9. Let S be a splitting submanifold of a complex manifold M. Given a first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$, let σ^* be the left splitting morphism associated to ρ. If $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$ has order of contact ν_f and l is such as in Remark 6.1.7 we shall denote by $\mathcal{F}^\sigma_f$ the coherent sheaf of $\mathcal{O}_S$-modules given by

$$\mathcal{F}^\sigma_f = \sigma^* \circ X_f \circ (df)^{\otimes \nu_f}(\mathrm{Sym}(\mathcal{N}_S)) \subset \mathcal{T}_S,$$

where $(df)^{\otimes \nu_f}$ is the endomorphism of $\mathrm{Sym}(\mathcal{N}_f)$ induced by the action of df on N_S. Notice that if $\nu_f > 1$ (or $\nu_f = 1$ and f tangential) we have that $df|_{N_S} = \mathrm{id}$ and hence the presence of df is meaningful only for $\nu_f = 1$ and f not tangential. We shall say that ρ is **f-faithful outside an analytic subset** $\Sigma \subset S$ if $\mathcal{F}^\sigma_f$ is the sheaf of germs of holomorphic sections of a sub-bundle of rank l of TS on $S \setminus \Sigma$. If $\Sigma = \emptyset$ we shall simply say that ρ is **f-faithful**.

One of the reasons we can apply our theory to the case of self maps is the following extension of a Lemma proved in [4], which is possible under some assumptions on the regularity of S.

Lemma 6.1.10. *Let $f \in End(M,S)$, $f \neq id_M$. Suppose S is a codimension m submanifold of M and the order of contact ν_f is larger than 1. Then $\mathcal{F}_f$ is a subbundle of $\mathcal{T}_{M,S(1)}$. Moreover, if f is tangential, $\mathcal{F}_f$ is a subbundle of $\mathcal{T}_{S(1)}$.*

Proof. We know from the proof of Lemma 6.1.8 that the coefficients of the canonical section are such that

$$(g_\alpha)^k_{r_1\dots r_{\nu_f}}\frac{\partial z^h_\beta}{\partial z^k_\alpha}=(h_{\alpha\beta})^{s_1}_{r_1}\cdots(h_{\alpha\beta})^{s_{\nu_f}}_{r_{\nu_f}}(g_\beta)^k_{s_1\dots s_{\nu_f}}+R_{\nu_f}. \tag{6.1}$$

We define now

$$v_{r_1\dots r_{\nu_f},\alpha}=[(g_\alpha)^k_{r_1\dots r_{\nu_f}}]_2\frac{\partial}{\partial z^k_\alpha}.$$

We want to show that there exists a cocycle $[c_{\alpha\beta}]_2$ such that

$$v_{r_1\dots r_{\nu_f},\alpha}=([c_{\alpha\beta}]_2)^{s_1\dots s_{\nu_f}}_{r_1\dots r_{\nu_f}}v_{s_1\dots s_{\nu_f},\beta}.$$

From equation (6.1) we see that indeed:

$$[(g_\alpha)^k_{r_1\dots r_{\nu_f}}]_2\left[\frac{\partial z^h_\beta}{\partial z^k_\alpha}\right]_2=\left[(h_{\alpha\beta})^{s_1}_{r_1}\cdots(h_{\alpha\beta})^{s_{\nu_f}}_{r_{\nu_f}}\right]_2[(g_\beta)^k_{s_1\dots s_{\nu_f}}]_2$$

and that

$$v_{r_1\dots r_{\nu_f},\alpha}=\left[(h_{\alpha\beta})^{s_1}_{r_1}\cdots(h_{\alpha\beta})^{s_{\nu_f}}_{r_{\nu_f}}\right]_2 v_{s_1\dots s_{\nu_f},\beta}.$$

If f is tangential, moreover, for each $r=1,\dots,m$ and each α we have that $(g_\alpha)^r_{r_1,\dots,r_{\nu_f}}$ belongs to $\mathcal{I}_S$; therefore, from Remark 3.3.2 we get that each of the $v_{r_1\dots r_{\nu_f},\alpha}$ is in $\mathcal{T}_{S(1)}$ and this proves the assertion. □

Remark 6.1.11. Please note that the canonical distribution is not, in general, involutive.

6.2 The results of ABT about holomorphic self maps

In paper [1] it was first proved a Camacho-Sad type residue theorem for holomorphic self maps tangent to the identity. The methods developed there were extended in the papers [3], [4]. In this section we will give a quick survey of the results in section 8 of [4], regarding the partial holomorphic connection on the normal bundle N_S arising from holomorphic self-maps. In Section 6.1 we gave the definition of canonical section and in Section 5.3 we showed how it is possible to get partial holomorphic connections for N_S looking for lifts of foliations into the Atiyah sheaf of S (Definition 5.3.1).

Remark 6.2.1. In the non tangential case, if there exists a f-faithful splitting, we shall put $v^\sigma_{r_1\dots r_{\nu_f},\alpha}=\sigma^*(v_{r_1\dots r_{\nu_f},\alpha})$ when $\nu_f>1$ and $v^\sigma_{r,\alpha}=\sigma^*\circ X\circ df(\partial_{r,\alpha})$ when $\nu_f=1$. To be consistent with the non-tangential case, even in the tangential case we shall put $v^\sigma_{r_1\dots r_{\nu_f},\alpha}:=v_{r_1\dots r_{\nu_f},\alpha}$, set $\sigma^*=\mathrm{id}$ and set $\mathcal{F}^\sigma_f:=\mathcal{F}_f$.

Proposition 6.2.2 ([4] Proposition 8.8 (first part))**.** *Let S be a codimension m submanifold of an n-dimensional complex manifold M. Let $f\in End(M,S)$,*

$f \not\equiv id_M$. If f is tangential, or $\nu_f > 1$ and there exists an f-faithful left splitting morphism, let $\{m_{\beta\alpha}\}$ be the cocycle defined by

$$m_{\beta\alpha} = \frac{\partial z_\beta^q}{\partial z_\alpha^p} \frac{\partial^2 z_\alpha^{s_1}}{\partial z_\beta^q \partial z_\beta^{s_2}} \frac{\partial z_\beta^{s_2}}{\partial z_\alpha^{s_3}} (g_\alpha)^p_{r_1 \dots r_{\nu_f}} \Big|_S \omega_\alpha^{s_3} \otimes \partial_{s_1,\alpha} \otimes v^{\sigma, r_1 \dots r_{\nu_f}, \alpha}.$$

We denote by $\mathfrak{m} \in H^1(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes \mathcal{F}_f^\sigma)$ the corresponding cohomology class. Then there exists a morphism $\phi : \mathcal{F}_f^\sigma \to \mathcal{A}$ such that $\Theta_1 \circ \phi = id$ if and only if $\mathfrak{m} = 0$.

Proposition 6.2.3 ([4] Proposition 8.8 (second part))**.** *Let S be a codimension m submanifold of an n-dimensional complex manifold M. Let $f \in End(M, S)$, $f \not\equiv id_M$. If f is not tangential and $\nu_f = 1$, let $\{m_{\beta\alpha}\}$ be the cocycle defined by*

$$m_{\beta\alpha} = \frac{\partial z_\beta^q}{\partial z_\alpha^p} \frac{\partial^2 z_\alpha^{s_1}}{\partial z_\beta^q \partial z_\beta^{s_2}} \frac{\partial z_\beta^{s_2}}{\partial z_\alpha^{s_3}} (\delta_{r_1}^{r_2} + (g_\alpha)^{r_2}_{r_1})(g_\alpha)^p_{r_2} \Big|_S \omega_\alpha^{s_3} \otimes \partial_{s_1,\alpha} \otimes v^{\sigma, r_1, \alpha}.$$

We denote by $\mathfrak{m} \in H^1(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes \mathcal{F}_f^\sigma)$ the corresponding cohomology class. Then there exists a morphism $\phi : \mathcal{F}_f^\sigma \to \mathcal{A}$ such that $\Theta_1 \circ \phi = id$ if and only if $\mathfrak{m} = 0$.

Now, the striking fact of paper [3] is if codimension S is 1, the vanishing of these cohomological classes is understood.

Proposition 6.2.4 ([4] Proposition 8.9 (tangential case))**.** *Let S be a codimension 1 submanifold of an n-dimensional complex manifold M. Let $f \in End(M, S)$, $f \not\equiv id_M$. If f is tangential then the cohomology class $\mathfrak{m} = 0$.*

Proposition 6.2.5 ([4] Proposition 8.9 (transverse case))**.** *Let S be a codimension 1 comfortably embedded submanifold of an n-dimensional complex manifold M. Let $f \in End(M, S)$, $f \not\equiv id_M$. Then the cohomology class $\mathfrak{m} = 0$.*

6.3 The variation action for holomorphic self maps

Thanks to Lemma 6.1.10 we know that in the tangential case, when the order of contact is higher than one, we can apply directly the theory we developed for foliations of the first infinitesimal neighborhood. The following theorem follows directly from Theorem 5.7.2.

Theorem 6.3.1. *Let S be a compact codimension m submanifold of M an n dimensional complex manifold. Let $f \in End(M, S)$, $f \neq id_M$. Suppose $\nu_f > 1$ and $\mathcal{F}_f$ is tangential to S. Let now $\mathcal{G}$ be the involutive closure of $\mathcal{F}_f$ on S and $\Sigma = S(\mathcal{F}_f) \cup S(\mathcal{G})$; let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be its decomposition in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f$ and $\mathcal{N}_{\mathcal{G},M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{G},M}),$$

where $\phi(\mathcal{N}_{\mathcal{G},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{G},M}$.

In the transverse case, when the order of contact is larger than one we can use the methods developed for 2-splitting submanifolds.

Theorem 6.3.2. *Let S be a compact codimension m submanifold 2-splitting in M, an n dimensional complex manifold. Let $f \in End(M,S)$, $f \neq id_M$, $\nu_f > 1$. Suppose we have a splitting $\mathcal{F}_f$-faithful outside an analytic subset of S containing $S(\mathcal{F}_f)$. Let now $\mathcal{G}$ be the involutive closure of $\mathcal{F}_f^\sigma$ and $\Sigma = S(\mathcal{F}_f^\sigma) \cup S(\mathcal{G})$; let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be its decomposition in connected components. Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) \in H_{2(n-m-k)}(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f^\sigma$ and $\mathcal{N}_{\mathcal{G},M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{G},M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{G},M}),$$

where $\phi(\mathcal{N}_{\mathcal{G},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{G},M}$.

These two theorems, even if their statement is really general, become much more interesting if the rank of $\mathcal{F}_f$ is 1, i.e., if S is a (smooth) hypersurface of M. In this case we have that $\mathcal{F}_f$ is a rank 1 foliation of S and our theory allows us to prove stronger results.

Theorem 6.3.3. *Let S be a (smooth) compact hypersurface of M an n dimensional complex manifold. Let $f \in End(M,S)$, $f \neq id_M$. Suppose $\nu_f > 1$ and $\mathcal{F}_f$ is tangential to S. Let $\Sigma = S(\mathcal{F}_f)$ and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be its decomposition in connected components. Then for every symmetric homogeneous polynomial ϕ of degree $n-1$ we can define the residue $Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{F}_f,M}; \Sigma_\lambda) \in H_0(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f$ and $\mathcal{N}_{\mathcal{F}_f,M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{F}_f,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F}_f,M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}_f,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}_f,M}$.

Theorem 6.3.4. *Let S be a (smooth) compact hypersurface of M an n dimensional complex manifold. Let $f \in End(M,S)$, $f \neq id_M$. Suppose $\nu_f > 1$ and suppose we have a splitting $\mathcal{F}_f$-faithful outside an analytic subset Σ of S containing $S(\mathcal{F}_f)$. Let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be its decomposition in connected components. Then for every symmetric homogeneous polynomial ϕ of degree $n-1$ we can define the residue $Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{F}_f,M}; \Sigma_\lambda) \in H_0(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f$ and $\mathcal{N}_{\mathcal{F}_f,M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f, \mathcal{N}_{\mathcal{F}_f,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F}_f,M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}_f,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}_f,M}$.

In case S is a splitting (smooth) hypersurface we can apply the results in Section 4.4, regardless of the order of contact, obtaining the following.

Theorem 6.3.5. *Let S be compact regular hypersurface splitting in M, an n dimensional complex manifold, and let $f \in End(M,S)$, $f \neq id_M$ and suppose* *there exists an $\mathcal{F}_f$-faithful splitting outside an analytic subset containing $S(\mathcal{F}_f)$. Then $\mathcal{F}_f^\sigma$ is a rank 1 holomorphic foliation defined on S. Suppose sequence (4.6) splits and let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. Then for every symmetric homogeneous polynomial ϕ of degree $n-1$ we can define the residue $Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{F}_f,M}; \Sigma_\lambda) \in H_0(\Sigma_\alpha)$ depending only on the local behaviour of $\mathcal{F}_f^\sigma$ and $\mathcal{N}_{F_f,M}$ near Σ_λ so that*

$$\sum_\lambda Res_\phi(\mathcal{F}_f^\sigma, \mathcal{N}_{\mathcal{F}_f^\sigma,M}; \Sigma_\lambda) = \int_S \phi(\mathcal{N}_{\mathcal{F}_f^\sigma,M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}_f^\sigma,M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}_f^\sigma,M}$.

6.4 Computing the variation index

Let S be a codimension 1 submanifold of a complex surface M and suppose there exists an f tangential, such that $\nu_f \geq 2$ and the associated canonical distribution has an isolated singularity in 0. The computation of the index follows almost directly from the one in section 5.5. First of all we remark that $l = \binom{1+2-1}{2} = 1$, so the canonical distribution if a rank 1 subbundle of $\mathcal{T}_{S(1)}$ with an isolated singularity in 0, i.e., a foliation of the first infinitesimal neighborhood outside 0, whose generator is given by

$$v = [g^1]_2 \frac{\partial}{\partial z^1} + [g^2]_2 \frac{\partial}{\partial z^2}.$$

Now, applying the formula developed for foliations of the first infinitesimal neighborhood we get that the residue for $c_1(N_{\mathcal{F}_f,M})$ in 0 is

$$\frac{1}{2\pi\sqrt{-1}} \int_{\{|y|=\varepsilon\}} \left[\frac{1}{g^2} \left(\frac{\partial g^1}{\partial x} + \frac{\partial g^2}{\partial y} \right) \right]_1 dy.$$

Suppose now we drop the tangentiality assumption and assume S is 2-splitting in M and we are working in an atlas adapted to the 2-splitting; the computation of the index follows directly from the one in Section 5.6. Again, $l = 1$ and $\mathcal{F}_f \subset \mathcal{T}_{M,S(1)}$; if the generator of $\mathcal{F}_f$ is given by

$$v = [g^1]_2 \frac{\partial}{\partial z^1} + [g^2]_2 \frac{\partial}{\partial z^2},$$

we know from our treatment that its projection is nothing else than

$$\sigma^*(v) = \left[\frac{\partial g^1}{\partial x} \cdot x \right]_2 \frac{\partial}{\partial z^1} + [g^2]_2 \frac{\partial}{\partial z^2}.$$

Now, the formula for the variation residue tells us that the residue for $c_1(N_{\mathcal{F}_f,M})$ in 0 is

$$\frac{1}{2\pi\sqrt{-1}} \int_{\{|y|=\varepsilon\}} \left[\frac{1}{g^2} \left(\frac{\partial g^1}{\partial x} + \frac{\partial g^2}{\partial y} \right) \right]_1 dy.$$

Chapter 7

Final Remarks

In this short section we would like to to pose some questions and to list what are in our opinion some interesting research paths.

The first thing we remark is that all the results of this thesis work only with **regular** submanifolds of a complex manifold M. It would be really interesting to generalize those results to the case first of local complete intersections and then to the case of general singular varieties.

We would like to stress that many of the results of this thesis are a starting point in a program towards the understanding of the following problem: when is it possible to extend a holomorphic foliation on a submanifold S of codimension m in a complex manifold M to a neighborhood of S? Thanks to Theorem 5.4.1 we know that, if there exists a rank l foliation of the first infinitesimal neighborhood, if we take any symmetric polynomial ϕ of degree larger than $n-m-l$ then $\phi(\mathcal{N}_{\mathcal{F},M})$ vanishes. Therefore, given a foliation $\mathcal{F}$ on S, the classes $\phi(\mathcal{N}_{\mathcal{F},M})$ are obstructions to find an extension to the first infinitesimal neighborhood, where ϕ is a symmetric polynomial of degree larger than $n-m-l$. In the splitting case we have much more information. As a matter of fact, if the sequence

$$0 \to \mathcal{V} \to \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}/\mathcal{V} \to 0$$

splits on the first infinitesimal neighborhood of the zero section of $\mathcal{N}_S$ we know that $\mathcal{F}$ can be extended in a non involutive way. Therefore, if S splits, the characteristic classes $\phi(\mathcal{N}_{\mathcal{F},M})$ with ϕ is a symmetric polynomial of degree larger than $n-m-l+\lfloor l/2 \rfloor$ are obstructions to find an extension of $\mathcal{F}$ as a not necessarily involutive subbundle of $\mathcal{T}_{S(1)}$. Known this, if the extension is involutive, also the characteristic classes $\phi(\mathcal{N}_{\mathcal{F},M})$ with ϕ a symmetric polynomial of degree larger $n-m-l$ and smaller than $n-m-l+\lfloor l/2 \rfloor$ vanish. Therefore, in the splitting case, known that there is a non-involutive extension, the classes $\phi(\mathcal{N}_{\mathcal{F},M})$ where ϕ is a symmetric polynomial of degree larger $n-m-l$ and smaller than $n-m-l+\lfloor l/2 \rfloor$ are obstructions to find an involutive extension. Now, it would be really interesting to see whether stronger conditions on the embedding, like k-linearizability, permit us to extend foliations, thanks to the vanishing of (4.6) to the k-th infinitesimal neighborhood of a submanifold and whether, thanks to some Grauert type results, it is possible to find an extension of the foliation to the whole ambient manifold.

Another important problem is to understand the dynamical meaning of the Khanedani-Lehmann-Suwa index, specially in the transversal case; as we said in

Remark 5.4.5 it could be connected with the intersection number of an algebraic leaf of the foliation and therefore it could be used in proving the non existence of algebraic leaves for a given foliation studying intersection numbers of algebraic leaves with target submanifolds.

Even if we have computed some of the residues associated to our index theorems there are still many computations to do. In particular case, it seems particularly interesting to compute the residues in the non involutive case and for involutive closures. As seen in Section 2.4 computing a residue is not elementary; this problem is strictly connected to that of finding examples of foliations and subsheaves of the infinitesimal tangent sheaf of S not arising from restriction of foliations and subsheaves of the ambient tangent manifold leaving S invariant.

In the preprint [2] Abate, Bracci, Tovena and Suwa have developed a theory for the localization of Atiyah classes; maybe some results of this thesis could be used in that framework.

I hope this thesis helped you understand the interest of the theory of indices and residues of singular holomorphic foliations and holomorphic self-maps. Still many things have to be done and understood but the beauty of the topic and its theoretical clarity made working on it a real pleasure.

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La luna nel cielo,
nelle mie suole il vento,
parole abusate
ho dato alla gioia,
parole abusate
ho dato al dolore,
parole abusate;
ho dato al sole
un nome segreto
che regalo solo
a chi voglio io.

Isaia